MATCHINGS, CONNECTIVITY, AND EIGENVALUES IN REGULAR GRAPHS

BY

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DISSERTATION

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Abstract

We study extremal and structural problems in regular graphs involving various parameters. In Chapter 2, we obtain the best lower bound for the matching number over n-vertex connected regular graphs in terms of edge-connectedness and determine when the matching number is minimized. We also establish the best upper bound for the number of cut-edges over n-vertex connected odd regular graphs and determine when the number of cut-edges is maximized. In addition, there is a relationship between the matching number and the total domination number in regular graphs. In Chapter 3, we explore the relationship between eigenvalue and matching number in regular graphs. We give a condition on an appropriate eigenvalue that guarantees a lower bound for the matching number of a l-edge-connected d-regular graph, when $l \leq d-2$. We also study what is the weakest hypothesis on the second largest eigenvalue λ_2 for a d-regular graph G to guarantee that G is l-edge-connected. In Chapter 4, we study several extremal problems for regular graphs, including the Chinese postman problem, the path cover number, the average edge-connectivity, and the number of perfect matchings. In Chapter 5, we study an r-dynamic coloring problem and give the relationship between the r-dynamic chromatic number and the chromatic number in regular graphs. We also study r-dynichromatic number of the cartesian product of paths and cycles.

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Chapter 1

Overview

In this thesis, we are interested in extremal problems for graphs, the relationship between eigenvalues and graph parameters, and structural graph parameters like connectivity. An extremal problem in graph theory asks for the maximum or minimum value of some parameter in terms of another parameter or over a family of graphs.

Part of why we study regular graphs is that they often arise when modeling real-life problems. For example, assume that you want to get married through a matching company, and you paid a fee to meet other people. Each person should meet the same number of people as others who paid the same fee. This condition corresponds to making the graph recording the meetings "regular".

As we study eigenvalues of a graph, we obtain information about other aspects, such as matchings or connectivity. In a communication network, we want to preserve network service by ensuring that the graph (or digraph) of possible transmission remains connected even when some vertices or edges fail. Connectivity measures how much we can delete. Bounds on graph eigenvalues can guarantee good connectivity properties.

1.1 Matchings in Regular Graphs

Matching theory involves existence problems (conditions for a perfect matching), enumeration problems (how many perfect matchings), and optimization problems (finding a maximum-sized matching).

A *matching* is a set of edges that pairwise share no vertices. A *perfect* matching in a graph G with n vertices is a matching consisting of $\frac{n}{2}$ edges. We denote by $\alpha'(G)$ the *matching number* of a graph G, which is the maximum size of a matching in G. A *cut-edge* of a connected graph G is an edge whose deletion leaves a disconnected subgraph. A graph G is *regular* if all vertices in $V(G)$ have the same degree; it is d-regular if every vertex has degree d.

In 1892, Petersen [53] proved that if a 3-regular graph has no cut-edges, then it has a perfect matching. A lot of questions were motivated by this result; we have studied several of them. A 3-regular graph having a cut-edge may have no perfect matching. We can ask whether there is a relationship between the existence of cut-edges and the existence of a perfect matching. In fact, Chartrand et.al. [17] gave a relationship between them in 1984. Let $c(G)$ be the number of cut-edges in a graph G.

Theorem 1.1.1. ([17]) *If G is an n-vertex connected cubic graph, then $\alpha'(G) \geq \frac{n}{2} - \frac{c(G)}{3}$.*

An upper bound on the number of cut-edges in connected n-vertex cubic graphs thus provides a lower bound on the matching number in them. In Section 2.1, we prove the following theorem.

Theorem 1.1.2. *If G is an n-vertex connected cubic graph, then $c(G) \leq \frac{n-7}{3}$.*

From these two theorems, we have $\frac{7n+14}{18}$ as a lower bound for the matching number of an n-vertex cubic graph. Now, we can ask whether the bound is the best lower bound for the matching number over n-vertex cubic graphs. More generally, we can consider this question for regular graphs of higher degree. Edge-connectivity becomes relevant, since for d-regular graphs with an even number of vertices, edge-connectivity $d-1$ forces a perfect matching. In general, we may ask how small the matching number of an l-edge-connected d-regular graph can be. The following theorems answer these questions and will appear in Section 2.1 and Section 2.2.

Theorem 1.1.3. *If G is a connected n-vertex $(2r+1)$-regular graph, then $\alpha'(G) \geq \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$.*

Theorem 1.1.4. *If G is a $(2t+1)$-edge-connected $(2r+1)$-regular graph with n vertices, where $0 \leq t \leq r$, then $\alpha'(G) \geq \frac{n}{2} - (\frac{r-t}{2(r+1)^2+t})\frac{n}{2}$.*

Theorem 1.1.5. *If G is a $2t$-edge-connected $2r$-regular graph with n vertices, where $1 \leq t \leq r$ and $r \geq 2$, then $\alpha'(G) \geq \frac{n}{2} - (\frac{r-t}{2r^2+r+t})\frac{n}{2}$.*

The lower bound when $t = 0$ in Theorem 1.1.4 gives a lower bound for the matching number over connected $(2r+1)$-regular graphs, but it is not as strong as the sharp result in Theorem 1.1.3

for that case. In fact, the proofs of the theorems are a little bit different. Characterizing when equality holds in an extremal problem provides insight into the structure of the graphs. In Section 2.2, we also characterize when equality holds in the conclusions of these theorems. These theorems also yield the minimum number of vertices of an l-edge-connected d-regular graph without a perfect matching. That problem was originally solved by Chartrand et al [16] for $l = d - 2$, and Niessen and Randerath [45] for general l.

We prove Theorem 1.1.3 using balloons. A *balloon* in a graph G is a maximal 2-edge-connected subgraph incident to exactly one cut-edge of G. Let $b(G)$ be the number of balloons. A vertex subset T of G is a total dominating set when every vertex in $V(G)$ has a neighbor in T. The total domination number of G, denoted $\gamma_t(G)$, is the minimum size of such a set.

Using balloons, we have an upper bound on $\gamma_t(G)$.

Theorem 1.1.6. *For a cubic graph G, $\gamma_t(G) \leq \frac{n}{2} - \frac{b(G)}{2}$ (except that $\gamma_t(G)$ may be $n/2 - 1$ when $b(G) = 3$ and the balloons cover all but one vertex).*

With $\alpha'(G) \geq \frac{n}{2} - \frac{b(G)}{3}$ for cubic graphs, this improves the known inequality $\gamma_t(G) \leq \alpha'(G)$.

To characterize when equality holds in Theorem 1.1.4 and Theorem 1.1.5, we introduce the notion of bullets in Section 2.2. Roughly speaking, the bullets $B_{r,t}$ and $B'_{r,t}$ are the smallest possible odd components left by deleting a smallest edge cut from a $(2r+1)$-regular graph with edge-connectivity $2t+1$ or from a $2r$-regular graph with edge-connectivity $2t$, respectively. Bullets and balloons are helpful to get a relationship between matching number and eigenvalues and we will show how to apply the notions to obtain the relationship in Chapter 3.

The Theorems proved in Chapter 2 are joint work with West and appear in [49], [50].

1.2 Edge-connectivity, Matching, and Eigenvalues in Regular Graphs

Eigenvalues are usually introduced in the context of matrix theory, but in mathematics history, they were derived from the study of quadratic forms and differential equations. A scalar λ is an *eigenvalue* of a square matrix A if there exists a nonzero vector x such that $Ax = \lambda x$. The *adjacency* *matrix* of a graph G, written $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$. The *Laplacian matrix* of a graph G is $D(G) - A(G)$, where $D(G)$ is the diagonal matrix of degrees and $A(G)$ is the adjacency matrix. The (ordinary) *eigenvalues* of a graph G are the eigenvalues of its adjacency matrix $A(G)$, and similarly, the *Laplacian eigenvalues* of a graph G are the eigenvalues of its Laplacian matrix.

A lot of research in graph theory over the last 40 years was stimulated by a classical result of Fiedler [25], stating that $\kappa(G) \geq \mu_2(G)$ for a non-complete graph G, where $\kappa(G)$ is the connectivity of G and $\mu_2(G)$ is the second smallest eigenvalue of the Laplacian matrix. In 2005, Haemers [28] found sufficient conditions on the Laplacian eigenvalues of a graph and on the third largest ordinary eigenvalue of a regular graph to guarantee the existence of a perfect matching.

Theorem 1.2.1. (Haemers [28]) *If G is a $2n$-vertex connected d-regular graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n}$ such that*

$$\lambda_3 \leq \begin{cases} d - 1 + \frac{3}{d+1} & \text{if } d \text{ is even,} \\ d - 1 + \frac{3}{d+2} & \text{if } d \text{ is odd,} \end{cases}$$

then G has a perfect matching.

It is natural to wonder why the relevant eigenvalue is the third largest eigenvalue and where the bounds in Theorem 1.2.1 come from. If a regular graphs G has no perfect matching and S is a subset with maximum deficiency in G, then the number of odd components in $G - S$ is at least 3. This 3 is the reason we look at the third largest eigenvalue of G. The odd components are related to balloons and bullets, and the numbers in Theorem 1.2.1 arise from the average degree in such graphs having $d - 2$ edges to S.

These bounds were improved by Cioabă and Gregory [14]. In 2009, Cioabă, Gregory, and Haemers [15] found the best possible conditions on the eigenvalues of a d-regular graph to guarantee the existence of a perfect matching. Recently, Cioabă and I generalized their result by giving a condition on an appropriate eigenvalue that guarantees a lower bound for the matching number of a l-edge-connected d-regular graph, when $l \leq d - 2$. The result of Cioabă, Gregory and Haemers is the special case $l = 1$ or $l = 2$ of our new result.

Theorem 1.2.2. *Denote by θ the greatest solution of the equation $x^3 - x^2 - 6x + 2 = 0$, and let*

$$\rho(d) = \begin{cases} \theta & \text{if } d = 3 \\ \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \geq 4 \text{ is even} \\ \frac{d-3+\sqrt{(d+1)^2+16}}{2} & \text{if } d \geq 5 \text{ is odd.} \end{cases} \tag{1.1}$$

Let $p \geq 3$ be an integer. If G is a t-edge-connected d-regular graph such that $\lambda_p(G) < \rho(d)$, then

$$\alpha'(G) > \begin{cases} \frac{n-p+\lfloor \frac{tp}{d} \rfloor}{2} & \text{when } d \equiv t \pmod 2 \\ \frac{n-p+\lfloor \frac{(t+1)p}{d} \rfloor}{2} & \text{when } d \equiv t+1 \pmod 2. \end{cases}$$

In fact, $\rho(3)$ is the largest eigenvalues of B_1, where B_1 is the smallest balloon in a cubic graph, and for $d \geq 4$, $\rho(d)$ is the largest eigenvalue of $B_{r,t}$ or $B'_{r,t}$, where $B_{r,t}$ and $B'_{r,t}$ are bullets in $(2t+1)$-edge-connected $(2r+1)$-regular graphs and a $2t$-edge-connected $2r$-regular graphs, respectively.

In Section 3.2, we also study the relationships between the eigenvalues of a d-regular t-edge-connected graph G and the maximum number of pairwise disjoint connected subgraphs in G that are each joined to the rest of the graph by exactly t edges.

Now, consider the graph $H_{r,t}$ obtained from two copies of $B_{r,t}$ or $B'_{r,t}$ by adding $2t+1$ edges or $2t$ edges between the two copies, respectively.

We can ask what is the weakest hypothesis on the second largest eigenvalue λ_2 for a d-regular graph G to guarantee that G is l-edge-connected. Cioaba [12] proved that if $\lambda_2 < d - \frac{2(l-1)}{d+1}$, then G is l-edge-connected. However, this result is not sharp; he proved stronger results when l is equal to 2 or 3.

Theorem 1.2.3. (Cioaba [12]) *Let d be an odd integer at least 3 and let $\pi(d)$ be the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If G is a d-regular graph such that $\lambda_2 < \pi(d)$, then $\kappa'(G) \geq 2$.*

Theorem 1.2.4. (Cioaba [12]) *If G is d-regular graph such that $\lambda_2(G) < \frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then $\kappa'(G) \geq 3$.*

Interestingly, the sharpness examples are derived from combining two copies of graphs in $H_{r,t}$.

In section 3.2, we also conjecture some open questions for general l.

Conjecture 1.2.5. *If G is a d-regular graph such that $\lambda_2(G) < \frac{d-4+\sqrt{(d+4)^2-8l}}{2}$ when d is odd and $\lambda_2(G) < \frac{d-3+\sqrt{(d+3)^2-8l}}{2}$ when d is even, then $\kappa(G)' \geq l+1$.*

In Section 3.2, we prove a partial positive answer to the Conjecture 1.2.5.

Some of the results of Chapter 3 are joint work with Cioaba and appear in [19]. The latter results will appear in [46].

1.3 Extremal Problems for Regular Graphs

Earlier we characterized the graphs achieving equality in our bounds on the matching number for connected regular graphs. These graphs in the family $\mathcal{H}_r'$ are also useful for studying the Chinese Postman Problem and the path covering number. The Chinese Postman Problem was introduced in the early 1960s by the Chinese mathematician Guan Meigu. Roughly speaking, a postman wishes to travel along every road in a city in order to deliver letters, with the least possible total distance. More precisely, a *postman tour* in a connected graph G is a closed walk containing all the edges of G. In a $(2r+1)$-regular graph, the problem is equivalent to finding a smallest spanning subgraph in which all vertices have odd degree. Let $p(G)$ be the minimum number of edges in a parity subgraph of G, where a *parity subgraph* is a spanning subgraph H of G such that $d_G(v) \equiv d_H(v) \pmod 2$ for every vertex v in G.

First, we determine the parity number of graphs in $\mathcal{H}_r'$.

Proposition 1.3.1. *If $G \in \mathcal{H}_r'$, then $p(G) = \frac{(2r^2+3r-1)n-2(r+1)}{4r^2+4r-2} - 1$, which reduces to $\frac{2n-5}{3}$ for cubic graphs.*

We are trying to establish a sharp bound for the solution in regular graphs of odd degree. We have done this for 3-regular graphs, where equality holds for only graphs in $\mathcal{H}_1'$. We also conjecture that if G is an n-vertex connected $(2r+1)$-regular graph, then $p(G)$ is bounded above by the parity number of graphs in $\mathcal{H}_r'$.

The results in Section 4.1 are joint work with West and appear in [51].

A *path cover* of a graph G is a vertex partition $V_1, \ldots, V_k$ such that every vertex in $V(G)$ belongs to exactly one of V_i and $G[V_i]$ has a spanning path for all i. The *path cover number* $q(G)$ is the minimum size of such a partition. In Section 4.2, we seek the best upper bound on $q(G)$ when G is r-regular.

In 1996, Reed proved that if G is an n-vertex 3-regular graph, then $q(G) \leq \lceil \frac{n}{9} \rceil$. Interestingly, the 3-regular graphs in the family $\mathcal{H}_r$ for $r = 1$ also achieve equality here. We also determine the parity number of graphs in the families $\mathcal{H}_r$, $\mathcal{H}_{r,t}$, and $\mathcal{H}'_{r,t}$ in Section 4.2.

Theorem 1.3.2. *If G is an n-vertex graph in $\mathcal{H}_r$, then $q(G) = \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1}$.*
If G is an n-vertex graph in $\mathcal{H}_{r,t}$, then $q(G) = \frac{(r-t)n}{2(r+1)^2+t}$.
If G is an n-vertex graph in $\mathcal{H}'_{r,t}$, then $q(G) = \frac{(r-t)n}{2r^2+r+t}$.

Furthermore, we believe that the following conjecture is true.

Conjecture 1.3.3. *If G is a graph in $\mathcal{F}_r$, then $q(G) \leq \lceil \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1} \rceil$.*
If G is a graph in $\mathcal{F}_{n,r,t}$, then $q(G) \leq \lceil \frac{(r-t)n}{2(r+1)^2+t} \rceil$.
If G is a graph in $\mathcal{F}'_{n,r,t}$, then $q(G) \leq \lceil \frac{(r-t)n}{2r^2+r+t} \rceil$.

In Section 4.2, we also give an upper bound for the path covering number over n-vertex 4-regular graphs, which may not be sharp. It remains to determine whether this bound is sharp.

The results in Section 4.2 appear in [47].

The connectivity and edge-connectivity of a graph measure how many edges must be deleted to disconnect the graph. However, since these values are based on a worst-case situation, it does not reflect the global connectedness of the graph. To measure the global connectedness of a graph G, we introduce *the average connectivity* of G.

The average connectivity of a graph G with n vertices, witten $\overline{\kappa}(G)$, is defined to be $\sum_{u,v \in V(G)} \frac{\kappa(u,v)}{\binom{n}{2}}$, where $\kappa(u,v)$ is the minimum number of vertices whose deletion makes v unreachable from u. By Menger's Theorem, this is equal to the minimum number of internally disjoint paths between u and v. Note that $\overline{\kappa}(G) \geq \kappa(G) = \min_{u,v \in V(G)} \kappa(u,v)$.

Similarly, we define *the average edge connectivity.*

The average edge-connectivity of a graph G with n vertices, witten $\overline{\kappa}'(G)$, is defined to be $\sum_{u,v \in V(G)} \kappa'(u,v)/\binom{n}{2}$, where $\kappa'(u,v)$ is the minimum number of edges whose deletion makes v

unreachable from u, which is same as the number of edge-disjoint pathes between u, v. Note that $\overline{\kappa}'(G) \geq \kappa'(G) = \min_{u,v\in V(G)} \kappa'(u,v)$.

In Section 4.3, we introduce some theorems of average connectivity which can be also applied to average edge connectivity and prove a relation between average connectivity and matching number.

Theorem 1.3.4. *For a connected graph G,*

$$\overline{k}(G) \leq 2\alpha'(G)$$

If G is a connected bipartite graph, then

$$\overline{k}(G) \leq \frac{9}{8}\alpha'(G) - \frac{3n-4}{8n^2-8n}\alpha'(G)$$

Even if you replace $\overline{\kappa}(G)$ with $\overline{\kappa}'(G)$in Theorem 1.3.4, then the theorem is still true.

Also, we prove a lower bound for the minimum value of the average edge connectivity of a connected regular graph with n vertices. It is sharp for infinitely many n and we characterize when equality holds in the bound.

Theorem 1.3.5. *If G is a connected cubic graph G with n vertices, which is not K_4, then*

$$\kappa'(G)\binom{n}{2} \geq \binom{n}{2} + \frac{7n+58}{4}.$$

We conjecture some open questions for odd regular graphs of higher degree.

As we have mentioned, every 3-regular graph without cut-edges has a perfect matching. Thus, it is natural to ask how many perfect matchings a 2-edge-connected 3-regular graph must have. In the 1970s, Lovász and Plummer conjectured that a 2-edge-connected 3-regular graph with n vertices has at least exponentially many (in n) perfect matchings. Voorhoeve [59] and Chudnovsky and Seymour [20] proved that the conjecture is true for bipartite graphs and planar graphs, respectively. Recently, Esperet, Kardos, King, Král, and Norine [23] proved the conjecture. 2-edge-connectedness forces a 3-regular graph to have a perfect matching. If we weaken the condition

"2-edge-connectedness" to "has a perfect matching", then how many perfect matchings must a 3-regular graph have? In Section 4.3, I answer this question, proving that for all even n with $n > 4$ the minimum number is only 4. There is an infinite sequence of 3-regular graphs having exactly 4 perfect matchings.

We conjecture some open questions for odd regular graphs of higher degree.

1.4 r-dynamic Coloring of Graphs

A teacher makes the following assignment: Each student must choose a country to study and explain to his or her friends. Each student with at least r friends must hear from friends about r different countries. A student with fewer friends must hear about different countries from all friends. In both cases, no two friends can study the same country. The students can plan together who will study which country. How many countries are needed? This motivates a coloring parameter. A *proper k coloring* of a graph G is a map f from $V(G)$ to S such that (i) S is a set of size k and (ii) if x and y are adjacent, then $f(x)$ and $f(y)$ are not equal. An *r-dynamic proper k-coloring* of a graph G is a proper k-coloring of G such that on the neighborhood of any vertex v, at least $\min\{r, d(v)\}$ distinct colors are used. The *r-dynamic chromatic number of a graph G*, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. Thus, $\chi_r(G)$ is the number of countries the students need. Montgomery introduced the notion of dynamic chromatic number in his dissertation [40]; he conjectured that if G is a regular graph, then $\chi_2(G) \leq \chi(G) + 2$, which is still open. The conjecture is true for bipartite graphs. In general, it is true that $\chi_2(G) \leq 2\chi(G)$. In Chapter 5, we prove that if G is a k-regular graph and $k \geq 7r\ln(r)$, then $\chi_r(G) \leq r\chi(G)$, where $\chi(G)$ is the chromatic number of G. In addition, we study the 2-dynamic chromatic number of a graph and the r-dynamic chromatic number of the cartesian product of two graphs. We are exploring other ways to improve upper bounds on $\chi_r(G)$. There have been a number of papers about 2-dynamic chromatic number, and r-dynamic chromatic number has studied as conditional chromatic number in a number of papers.

1.5 Background material

For completeness, here we present some basic definitions about graphs. A *graph* is a pair consisting of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge is an unordered pair of vertices called its *endpoints*. When u and v are the endpoints of an edge, two vertices u and v in $V(G)$ are *adjacent* in G and are *neighbors*. The number of vertices adjacent to v is the *degree* of v, written $d(v)$. A graph G is *regular* if all vertices in $V(G)$ have the same degree; it is d-regular if every vertex has degree d.

A vertex v and an edge e are *incident* if v is an endpoint of e. Also two edges e and f in $E(G)$ are *incident* in G if e and f have a common endpoint. A graph is *finite* if its vertex set and edge set are finite. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise nonadjacent vertices. A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite sets* of G. A graph G is *k-partite* if $V(G)$ can be expressed as the union of k (possibly empty) independent sets. A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted K_n. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes r and s, the complete biparite graph is denoted $K_{r,s}$. If H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we call H a *subgraph* of G. We then write $H \subseteq G$ and say that "G contains H". The *union* of graphs $G_1, \dots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

A *path* with n vertices is a graph whose vertex set can be indexed as $\{v_1, ..., v_n\}$ so that its edge set is $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$. A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A *walk* is a list $v_0, e_1, v_1, \dots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A *trail* is a walk with no repeated edge. A *u,v-walk* or *u,v-trail* has first vertex u and last vertex v; these are its *endpoints*.

A *u,v-path* is a path whose vertices of degree 1 (its *endpoints*) are u and v; the others are *internal vertices*. The *length* of a walk, trail, path, or cycle is its number of edges. A walk or trail is *closed* if its endpoints are the same. A walk or trail or path is *odd* or *even* as its length is odd or even. A graph G is *connected* if for all $u, v \in V(G)$, there is a path containing u and v. Otherwise, G is *disconnected*. The *components* of a graph G are its maximal connected subgraphs. A component (or graph) is *trivial* if it has no edges; otherwise it is *nontrivial*. An *isolated vertex* is a vertex of degree 0. A graph is *Eulerian* if it has a closed trail containing all edges. We call a closed trail a *circuit* when we do not specify the first vertex but keep the list in cyclic order. An *Eulerian circuit* or *Eulerian trail* in a graph is a circuit or trail containing all the edges. An *even graph* is a graph with vertex degrees all even. A vertex is it odd[even] when its degree is odd[even]. A *postman tour* in a connected graph G is a closed traversal of all the edges of G. When $\mathcal{P}$ is a set of disjoint-paths and every vertex in $V(G)$ belongs to exactly one path, we call $\mathcal{P}$ a *path cover* of G. The *path cover number* of G, denoted $p(G)$, is the minimum size of such a set.

A graph G is *k-connected* if it has more than k vertices and there is no set of $k-1$ vertices whose removal disconnects it. We denote by $\kappa(G)$ the *connectivity* of a graph, which is the largest k such that G is k-connected. Similarly, a graph G is *k-edge-connected* if there is no set of $k-1$ edges whose removal disconnects it. We denote by $\kappa'(G)$ the *edge-connectivity* of a graph, which is the largest k such that G is k-edge-connected. *The average connectivity* of a graph G with n vertices, witten $\overline{\kappa}(G)$, is defined to be $\sum_{u,v\in V(G)} \kappa(u,v)/\binom{n}{2}$, where $\kappa(u,v)$ is the minimum number of vertices whose deletion makes v unreachable from u. Note that $\overline{\kappa}(G) \geq \kappa(G) = \min_{u,v\in V(G)} \kappa(u,v)$. Similarly, *The average edge-connectivity* of a graph G with n vertices, witten $\overline{\kappa}'(G)$, is defined to be $\sum_{u,v\in V(G)} \kappa'(u,v)/\binom{n}{2}$, where $\kappa'(u,v)$ is the minimum number of edges whose deletion makes v unreachable from u. Note that $\overline{\kappa}'(G) \geq \kappa'(G) = \min_{u,v\in V(G)} \kappa'(u,v)$.

A *matching* is a set of edges that pairwise share no vertices. A *perfect matching* in a graph G with n vertices is a matching consisting of $\frac{n}{2}$ edges. We denote by $\alpha'(G)$ the *matching number* of a graph G, which is the maximum size of a matching in G. A *cut-edge* or *cut-vertex* of a connected graph G is an edge or vertex whose deletion leaves a disconnected subgraph. We write $G-e$ or $G-M$ for the subgraph of G obtained by deleting an edge e or set of edges M. We write $G-v$

or $G - S$ for the subgraph of G obtained by deleting a vertex v or set of vertices S. An *induced subgraph* is a subgraph obtained by deleting a set of vertices. We write $G[T]$ for $G - \overline{T}$, where $\overline{T} = V(G) - T$; this is the subgraph of G *induced by* T.

A vertex subset T of G is a *total dominating set* when every vertex in $V(G)$ has a neighbor in T. The *total domination number* of G, denoted $\gamma_t(G)$, is the minimum size of such a set.

A scalar λ is an *eigenvalue* of a square matrix A if there exists a nonzero vector x such that $Ax = \lambda x$. The *adjacency matrix* of a graph G, written $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$. If $a_{i,j} = a_{j,i}$ for all i and j, then A is *symmetric*. The *Laplacian matrix* of a graph G is $D(G) - A(G)$, where $D(G)$ is the diagonal matrix of degrees and $A(G)$ is the adjacency matrix. The (ordinary) *eigenvalues* of a graph G are the eigenvalues of its adjacency matrix $A(G)$, and similarly, the *Laplacian eigenvalues* of a graph G are the eigenvalues of its Laplacian matrix.

A *k-coloring* of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are *colors*; the vertices of one color form a *color class*. A k-coloring is *proper* if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable. An *r-dynamic proper k-coloring* of a graph G is a proper k-coloring of G such that every vertex in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The *r-dynamic chromatic number* of a graph G, written $\chi_r(G)$, is the minimum number k for which G has an r-dynamic proper k-coloring.

Chapter 2

Matchings in Regular Graphs

A *balloon* in a graph G is a maximal 2-edge-connected subgraph incident to exactly one cut-edge of G. Let $b(G)$ be the number of balloons, let $c(G)$ be the number of cut-edges, and let $\alpha'(G)$ be the maximum size of a matching. Let $\mathcal{F}_n$ be the family of connected $(2r+1)$-regular graphs with n vertices. In this section, for $G \in \mathcal{F}_n$, we prove $c(G) \leq \frac{r(n-2)-2}{2r^2+2r-1} - 1$ and $\alpha'(G) \geq \frac{n}{2} - \frac{rb(G)}{2r+1}$. Also $b(G) \leq \frac{(2r-1)n+2}{4r^2+4r-2}$, which yields a simple proof of the lower bound on $\alpha'(G)$ by Henning and Yeo (about $\frac{n}{2} - \frac{n}{4r}$ for large r). For each of these bounds and each r, we determine the infinite family where equality holds. For the total domination number $\gamma_t(G)$ of a cubic graph, we prove $\gamma_t(G) \leq \frac{n}{2} - \frac{b(G)}{2}$ (except that $\gamma_t(G)$ may be $n/2 - 1$ when $b(G) = 3$ and the balloons cover all but one vertex). With $\alpha'(G) \geq \frac{n}{2} - \frac{b(G)}{3}$ for cubic graphs, this improves the known inequality $\gamma_t(G) \leq \alpha'(G)$.

Henning and Yeo proved a lower bound for the minimum size of a maximum matching among connected k-regular graphs with n vertices; it is sharp infinitely often, and in Section 2.1, we characteriz when equality holds. In Section 2.2, we prove a lower bound for the minimum size of a maximum matching in a l-edge-connected k-regular graph with n vertices, for $l \geq 2$ and $k \geq 4$; it is sharp for infinitely many n. We also characterize when equality holds in the bound.

2.1 Balloons, Cut-edges, Matchings, and Total Domination

A graph is a *cubic graph* if every vertex has degree 3. In 1891, Petersen [53] proved that every cubic graph without cut-edges has a perfect matching. It is natural to ask how small $\alpha'(G)$ can be in a cubic graph G with n vertices, where $\alpha'(G)$ is the maximum size of a matching in G (called the *matching number* of G). Chartrand et al. [17] proved that $\alpha'(G) \geq n/2 - \lceil c(G)/3 \rceil$ when G is a cubic n-vertex graph, where $c(G)$ denotes the number of cut-edges in G.

By this result, an upper bound on $c(G)$ yields a lower bound on $\alpha'(G)$. Let G be a connected cubic graph with n vertices. In Section 3, we prove that $c(G) \le (n-7)/3$ and that this is sharp. The result of [17] then yields $\alpha'(G) \ge (7n+14)/18$, but this is not the best bound on $\alpha'(G)$. The smallest value of $\alpha'(G)$ is $\lceil (4n-1)/9 \rceil$, proved first by Biedl et al. [8]. Henning and Yeo [32] generalized the result, proving that $\alpha'(G) \ge \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n-1}{(2r+1)(2r^2+2r-1)}$ when G is a $(2r+1)$-regular n-vertex connected graph, which is sharp.

Although maximizing $c(G)$ in a cubic graph does not minimize $\alpha'(G)$, another concept does yield a simple proof of the sharp bound on $\alpha'(G)$. We define a *balloon* in a graph G to be a maximal 2-edge-connected subgraph of G incident to exactly one cut-edge of G. The term arises from viewing the cut-edge as a string tying the balloon to the rest of the graph; the vertex incident to the cut-edge is the *neck* of the balloon. A balloon may contain cut-vertices and thus consist of several blocks.

Maximal 2-edge-connected subgraphs are pairwise disjoint, since the union of two 2-edge-connected subgraphs sharing a vertex is also 2-edge-connected. Among these subgraphs, the balloons are those incident to precisely one cut-edge. Thus the number of balloons in G is well-defined; let $b(G)$ denote this number.

Let $\mathcal{F}_{n,r}$ be the family of connected $(2r+1)$-regular graphs with n vertices. For $G \in \mathcal{F}_{n,r}$, we prove that $c(G) \le \frac{r(n-2)-2}{2r^2+2r-1} - 1$ and $\alpha'(G) \ge \frac{n}{2} - \frac{rb(G)}{2r+1}$. We obtain a lower bound on $\alpha'(G)$ by proving that $b(G) \le \frac{(2r-1)n+2}{4r^2+4r-2}$, and we use balloons to prove the upper bound on $c(G)$. We construct an infinite family $\mathcal{H}_r$ showing that all these bounds are sharp; it contains the smaller families provided in [8] and [32] (graphs in $\mathcal{H}_r$ exist when $n \equiv 4(r+1)^2 \mod (8r^3+12r^2-2)$). The bounds for $b(G)$ and $c(G)$ are sharp in a larger family $\mathcal{H}'_r$ (occurring when $n \equiv (4r+6) \mod (4r^2+4r-2)$). We prove the upper bounds on $b(G)$ and $c(G)$ and show that equality holds if and only if $G \in \mathcal{H}'_r$. Subsequently, we also prove the lower bound on $\alpha'(G)$ and show that equality holds if and only if $G \in \mathcal{H}_r$.

The restriction to connected graphs is important; consider cubic graphs. For a connected cubic graph, $b(G) \le (n+2)/6$ and $\alpha'(G) \ge (4n-1)/9$. However, if G consists of many disjoint copies of the unique 16-vertex cubic graph with no perfect matching, then $b(G) = 3n/16$ and $\alpha'(G) = 7n/16$; these values are more extreme than the bounds for graphs in $\mathcal{F}_{n,r}$.

We use balloons to study total domination. A *total dominating set* in a graph G is a set S of vertices in G such that every vertex in G has a neighbor in S. The *total domination number*, written $\gamma_t(G)$, is the minimum size of a total dominating set in G. Henning, Kang, Shan, and Yeo [29] proved that $\gamma_t(G) \le \alpha'(G)$ for every regular graph G with degree at least 3. For degree at least 4, stronger bounds hold. Thomassé and Yeo [58] proved that $\gamma_t(G) \le 3n/7$ for every n-vertex regular graph with degree at least 4. This upper bound is a smaller fraction of n than the lower bound on $\alpha'(G)$. Earlier, Henning and Yeo [32] observed that $\gamma_t(G) < \alpha'(G)$ when G is a regular graph with degree at least 4.

We use balloons to strengthen the bound for cubic graphs. We prove that $\gamma_t(G) \le \frac{n}{2} - \frac{b(G)}{2}$ when G is cubic, except that $\gamma_t(G) = n/2 - 1$ is possible when $b(G) = 3$ and the balloons cover all but one vertex. Since $\alpha'(G) \ge \frac{n}{2} - \frac{b(G)}{3}$ for cubic graphs, we have large separation when the number of balloons is large, and $\gamma_t(G) = \alpha'(G)$ can happen in a cubic graph only when the number of balloons is 0 or when G consists of three balloons plus one vertex.

We mention one related result. The extension of Petersen's result by Bäbler [6] states that every $(2r+1)$-regular $2r$-edge-connected graph has a perfect matching. As the edge-connectivity rises, the lower bound on the matching number should also rise. We solve this problem in a subsequent paper [50], determining the smallest matching number for d-regular k-edge-connected graphs with n vertices, when $d \ge 4$ and $k \ge 2$. The proof differs somewhat from the techniques in this paper, since k-edge-connected graphs have no balloons. A generalization of balloons is needed.

Biedl et al. [8] and Henning and Yeo [32] presented examples for sharpness in the lower bounds on $\alpha'(G)$ for connected 3-regular and $(2r+1)$-regular graphs, respectively. We present a more general family that includes their examples.

Construction 2.1.1. Let B_r be the graph obtained from the complete graph K_{2r+3} by deleting a matching of size $r+1$ and one more edge incident to the remaining vertex. This is the smallest graph in which one vertex has degree $2r$ and the others have degree $(2r+1)$. Thus B_r is the smallest possible balloon in a $(2r+1)$-regular graph. Note that deleting the vertex of degree $2r$

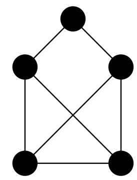

Figure 2.1: The smallest possible balloon in a cubic graph, B_1

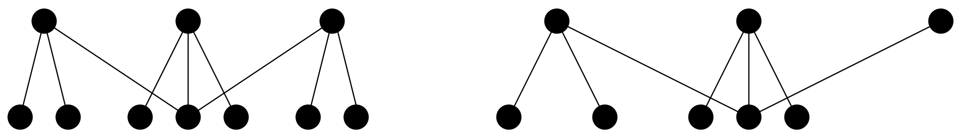

Figure 2.2: The graphs in $\mathcal{T}_1$ and $\mathcal{T}_1'$

(the neck) from B_r leaves a subgraph having a perfect matching.

Let $\mathcal{T}_r'$ be the family of trees such that every non-leaf vertex has degree $2r+1$. Let $\mathcal{H}_r'$ be the family of $(2r+1)$-regular graphs obtained from trees in $\mathcal{T}_r'$ by identifying each leaf of such a tree with the neck in a copy of B_r. Let $\mathcal{T}_r$ be the subfamily of $\mathcal{T}_r'$ obtained by requiring all leaves to have the same color in a proper 2-coloring. Let $\mathcal{H}_r$ be the subfamily of $\mathcal{H}_r'$ arising from trees in $\mathcal{T}_r$ by adding balloons at leaves. $\square$

Figure 2.1 describes B_1, and Figures 2.2 and 2.3 show the distinction between $\mathcal{T}_r$ and $\mathcal{T}_r'$, and $\mathcal{H}_r$ and $\mathcal{H}_r'$ when $r=1$.

To compute the matching number for n-vertex graphs in $\mathcal{H}_r$, we use standard concepts about matchings. The *deficiency* of a vertex set S in a graph G, written $\mathrm{def}_G(S)$ or simply $\mathrm{def}(S)$, is $o(G-S)-|S|$, where $o(H)$ is the number of components of H having an odd number of vertices.

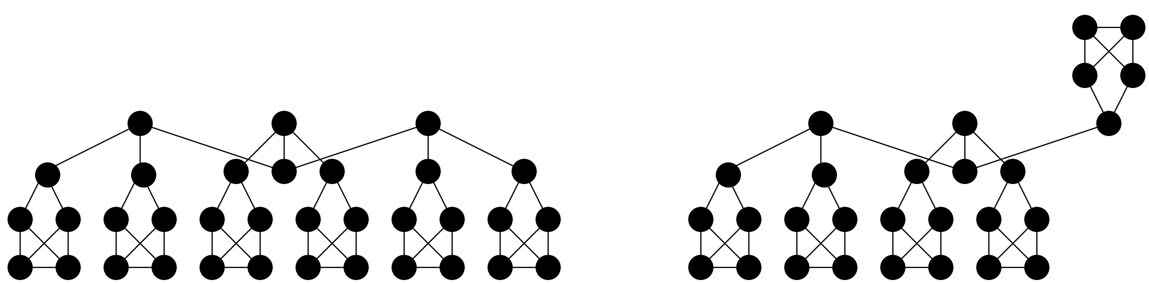

Figure 2.3: The graphs in $\mathcal{H}_1$ and $\mathcal{H}_1'$

Every matching must leave at least $\operatorname{def}(S)$ vertices unmatched, so for any S the quantity $\frac{1}{2}(n - \operatorname{def}(S))$ is an upper bound on $\alpha'(G)$. Furthermore, if there is a matching that matches S into vertices of distinct odd components of $G-S$ and leaves at most one unmatched vertex in each odd component of $G-S$, then $\alpha'(G) = \frac{1}{2}(n - \operatorname{def}(S))$.

Proposition 2.1.2. *Let $p_r = 2r^2 + 2r - 1$. For any n-vertex graph G in $\mathcal{H}_r$,*

$$\begin{aligned} n &\equiv 4(r+1)^2 \mod ((4r+2)p_r), & b(G) &= \tfrac{(2r-1)n+2}{2p_r}, \\ \alpha'(G) &= \tfrac{1}{2}\left(n - \tfrac{r(2r-1)n+2r}{(2r+1)p_r}\right), & c(G) &= \tfrac{r(n-2)-2}{p_r} - 1. \end{aligned}$$

Furthermore, the formulas given for $b(G)$ and $c(G)$ also hold when $G \in \mathcal{H}'_r$.

Proof. We first compute $b(G)$ and $c(G)$ on $\mathcal{H}'_r$. The smallest tree in $\mathcal{T}'_r$ has two vertices. The resulting graph in $\mathcal{H}'_r$ has $4r+6$ vertices, two balloons, and one cut-edge, and the formulas hold. For any larger tree T in $\mathcal{T}'_r$, the penultimate vertex of a longest path has $2r$ leaf neighbors, and deleting them yields a smaller tree T' in $\mathcal{T}'_r$. Let G and G' be the corresponding graphs in $\mathcal{H}'_r$. Compared to G', in G there are $2r$ more cut-edges, $2r-1$ more balloons, and $2r(2r+3)-(2r+2)$ more vertices. This last formula simplifies to $2p_r$, and hence the formulas for $b(G)$ and $c(G)$ in terms of n are established by induction on n.

Now consider the more restrictive families $\mathcal{T}_r$ and $\mathcal{H}_r$. The smallest graph in $\mathcal{T}_r$ is the star $K_{1,2r+1}$ with $2r+1$ leaves. We claim that every other tree in $\mathcal{T}_r$ arises from a smaller tree in $\mathcal{T}_r$ by appending $2r$ edges at a leaf y and appending $2r$ additional edges at each new neighbor of y. This produces $(2r)^2$ leaves, which replace y in the set of leaves and are in the same partite set as y, so the larger graph lies in $\mathcal{T}_r$.

To prove that this generates all of $\mathcal{T}_r$, consider a longest path P in a tree $T \in \mathcal{T}_r$ such that T is not a star. Let y, z, w be the last three vertices on P, in order (w is the leaf). Since P is a longest path, all $2r$ neighbors of z other than y are leaves. Since leaves all lie in the same partite set, no neighbor of y is a leaf. Hence the $2r-1$ neighbors of y not on P must all have $2r$ leaf neighbors (again since P is a longest path and non-leaves have degree $2r+1$). Now T arises in the specified way from a smaller tree in $\mathcal{T}_r$ having y as a leaf.

To compute $\alpha'(G)$ for $G \in \mathcal{H}_r$, let T be the corresponding tree in $\mathcal{T}_r$. Let X and Y be its partite sets, with Y containing the leaves. Let $S = X$. Now $o(G-S) = |Y|$, since each vertex of Y is an isolated vertex in $G-S$ or is the neck of a copy of B_r that is an odd component of $G-S$. Thus $\mathrm{def}(S) = |Y| - |X|$. Root T at a vertex of X, and then match each vertex of S to one of its children, which is or lies in an odd component of $G-S$. When that odd component is a copy of B_r, pair its remaining vertices in a matching. This produces a matching with exactly $\mathrm{def}(S)$ uncovered vertices.

It therefore suffices to compare $\mathrm{def}(S)$ and the formula for $\alpha'(G)$ inductively. When $T = K_{1,2r+1}$, we have $\mathrm{def}(S) = 2r$. Adding the balloons yields $(2r+3)(2r+1)+1$ (this equals $4(r+1)^2$, giving the basis for the claim about n). The subtractive term in the formula for $\alpha'(G)$ is $\frac{r(2r-1)(4r^2+8r+4)+2r}{(2r+1)p_r}$, which equals $2r$.

For larger $G \in \mathcal{H}_r$, let T be the corresponding tree in $\mathcal{T}_r$, expanded from T' with corresponding graph $G' \in \mathcal{H}_r$. In the expansion, $|X|$ increases by $2r$ and $|Y|$ increases by $4r^2$, so $\mathrm{def}(S)$ increases by $4r^2 - 2r$. Comparing G with G', one balloon is lost and $4r^2$ are created; the number of vertices increases by $4r^2(2r+3) + 2r - (2r+2)$. The increase in n simplifies to $(4r+2)p_r$ (completing the proof of the claim about n). The subtractive term in the formula for $\alpha'(G)$ thus increases by $r(2r-1)2$, which equals the change in $\mathrm{def}(S)$. □

Corollary 2.1.3. *For n-vertex cubic graphs, the matching number of graphs in $\mathcal{H}_1$ is $\frac{4n-1}{9}$.*

Recall that $\mathcal{F}_{n,r}$ is the family of connected $(2r+1)$-regular graphs with n vertices. We begin by bounding the number of balloons for graphs in $\mathcal{F}_{n,r}$.

Every balloon in a $(2r+1)$-regular graph has at least $2r+3$ vertices; it has at least $2r+2$ vertices because it has a vertex of degree $2r+1$, and equality cannot hold because then the degree-sum would be odd. Thus $b(G) \le \frac{n}{2r+3}$. Surprisingly, this trivial upper bound can be improved only slightly; the optimal bound is $\frac{n+2\epsilon}{2r+3+\epsilon}$, where $\epsilon = 1/(2r-1)$. Of course, $\epsilon = 1$ for cubic graphs. We use a counting argument; the bound can also be proved inductively.

Lemma 2.1.4. *If $G \in \mathcal{F}_{n,r}$, then $b(G) \le \frac{(2r-1)n+2}{4r^2+4r-2}$, with equality if and only if $G \in \mathcal{H}'_r$.*

Proof. For $G \in \mathcal{F}_{n,r}$, let G' be the graph obtained from G by shrinking each balloon to a single vertex; G' is connected, and the balloons of G become vertices of degree 1 in G'. Let $n' =$

$|V(G')|$ and $m' = |E(G')|$. Since G' is connected, $m' \geq n' - 1$, and the degree-sum formula yields $(2r+1)n' - 2rb(G) = 2m' \geq 2n' - 2$. Thus $2rb(G) \leq (2r-1)n' + 2$. Since each balloon has at least $2r+3$ vertices, $n' \leq n - (2r+2)b(G)$. Combining the inequalities yields $2rb(G) \leq (2r-1)n + 2 - (2r-1)(2r+2)b(G)$, which simplifies to the desired bound.

Equality requires equality in each contributing inequality. Hence G' is a tree with non-leaf vertices having degree $2r+1$. That is, $G' \in \mathcal{T}'_r$, and $G \in \mathcal{H}'_r$. $\square$

Corollary 2.1.5. *Every connected n-vertex cubic graph has at most $\frac{n+2}{6}$ balloons, and this is sharp for $n \equiv 4 \mod 6$.*

The bounds of Lemma 2.1.4 and Corollary 2.1.5 do not hold for disconnected graphs. An n-vertex graph consisting of disjoint copies of the smallest graph in $\mathcal{H}_r$ has $\frac{2r+1}{6r+10}n$ balloons, which is more than the bound above.

Lemma 2.1.6. *The following hold for balloons and cut-edges in graphs in $\mathcal{F}_{n,r}$.*

(a) Each component formed by deleting a cut-edge contains a balloon.

(b) Balloons may have any odd number of vertices at least $2r+3$.

Proof. (a) Let e be a cut-edge. Among the paths containing e, let P be a path containing the maximum number of cut-edges of G. The portion of P after the last cut-edge toward either end lies in a 2-edge-connected subgraph, and by the choice of P it is a balloon.

(b) In a balloon, the neck has degree $2r$, and other vertices have degree $2r+1$. Such graphs exist with every odd number of vertices at least $2r+3$. For $k \geq r$, the complete graph K_{2k+3} decomposes into $k+1$ spanning cycles. The union of r of these cycles plus a near-perfect matching from one of the remaining cycles is a 2-edge-connected graph with the desired degrees. $\square$

Lemma 2.1.7. *If $G \in \mathcal{F}_{n,r}$, then $c(G) \leq \frac{r(n-2)-2}{2r^2+2r-1} - 1$, with equality if and only if $G \in \mathcal{H}'_r$.*

Proof. We use induction on n. If $n \leq 4r+6$, then the bound at most 1, with equality only when $n = 4r+6$. Every graph having a cut-edge has at least two balloons and hence at least $4r+6$ vertices, by Lemma 2.1.6. The graph with $4r+6$ vertices consisting of two copies of B_r joined by an edge lies in $\mathcal{H}'_r$. Hence all claims hold for the basis.

For larger n, consider a cut-edge e in G. Let G_1 and G_2 be the components of $G-e$. Let G_1' and G_2' be the graphs obtained from G by replacing G_2 and G_1, respectively, with B_r. The cut-edges of G consists of the cut-edges in G_1 and G_2, plus e itself. Since e is a cut-edge in both G_1' and G_2', and the added B_r contains no cut-edge, we have $c(G) = c(G_1') + c(G_2') - 1$. If neither G_1 nor G_2 equals B_r, then G_1' and G_2' have fewer vertices than G, and we can apply the induction hypothesis to both. Letting $n_i = |V(G_i')|$, we have $n = n_1 + n_2 - (4r+6)$. With $p_r = 2r^2 + 2r - 1$ (as in Proposition 2.1.2), we obtain the desired bound on $c(G)$:

$$\begin{aligned} c(G) &= c(G_1') + c(G_2') - 1 \ \le\ \frac{r(n_1-2)-2}{p_r} + \frac{r(n_2-2)-2}{p_r} - 3 \\ &= \frac{r(n-2)-2}{p_r} + \frac{r(4r+4)-2}{p_r} - 3 \ =\ \frac{r(n-2)-2}{p_r} - 1. \end{aligned}$$

In the remaining case, every cut-edge in G is incident to a copy of B_r. Since each copy of B_r is incident to exactly one cut-edge, we obtain $c(G) = b(G)$ (note that $n > 4r+6$). Let Q be the set of endpoints of cut-edges outside the balloons. If any two balloons have distinct nonadjacent neighbors in Q, then let G' be the graph obtained by deleting the two balloons and adding one edge to make their neighbors adjacent. The graph G' is connected and $(2r+1)$-regular and has $n-(4r+6)$ vertices. Crucially, G' has exactly $c(G)-2$ cut-edges, because the only cut-edges in G are those incident to balloons. By the induction hypothesis,

$$c(G) \le 2 + \frac{r(n-4r-8)-2}{p_r} - 1 = \frac{r(n-2)-2}{p_r} - \frac{4r^2+6r}{p_r} + 1 < \frac{r(n-2)-2}{p_r} - 1.$$

Hence we may assume that the vertices of Q are pairwise adjacent. Let $q = |Q|$, and let S be the set of vertices outside both Q and the balloons. If $S = \varnothing$, then $c(G) = q(2r+2-q)$ and $n = (2r+3)c(G) + q$. Since $1 \le q \le 2r+1$, we obtain $n \ge (2r+3)(2r+1) + 1 = 2p_r + 4r + 6$. Since $c(G) = b(G)$, Lemma 2.1.4 yields $c(G) \le \frac{(2r-1)n+2}{2p_r} = \frac{rn}{p_r} - \frac{n-2}{2p_r}$. It thus suffices to show that $\frac{n-2}{2p_r} \ge \frac{2r+2}{p_r} + 1$. This requires $n - 2 \ge 4r + 4 + 2p_r$, which we have proved for this case.

Finally, suppose that $S \ne \varnothing$. Each vertex of S has $2r+1$ neighbors outside the balloons, so $n \ge 2r + 2 + (2r+3)c(G)$. If equality holds, then $S \cup Q$ induces a complete graph, $G = K_{r+2}$, and $c(G) = 0$. Otherwise, $n \ge (2r+3)[c(G)+1]$. Now $c(G) \le \frac{n}{2r+3} - 1$, and we only need

$\frac{n}{2r+3} \le \frac{r(n-2)-2}{2r^2+2r-1}$. This simplifies to $n \ge 4r+6$, which holds when $c(G) > 0$.

For the characterization of equality, consider each case. When G has a cut-edge not incident to a balloon that is a copy of B_r, the induction hypothesis requires achieving equality for both G_1' and G_2', which must therefore lie in $\mathcal{H}_r'$. The construction of G from G_1' and G_2' indeed puts G in $\mathcal{H}_r'$. When $c(G) = b(G)$ and two balloons have nonadjacent neighbors, we obtained strict inequality in the bound. When $c(G) = b(G)$ and $S = \varnothing$, equality requires $b(G)$ to meet its bound, which already requires $G \in \mathcal{H}_r'$ (indeed, it requires more, and equality is obtained only by putting copies of B_r at the leaves of the star $K_{1,2r+1}$). When $S \ne \varnothing$, equality requires $n = 4r+6$ and $c(G) = 1$, in which case G is the graph in $\mathcal{H}_r'$ consisting of a cut-edge joining two copies of B_r. $\square$

Corollary 2.1.8. *Every n-vertex $(2r+1)$-regular graph has at most $\frac{r(n-2)-2}{2r^2+2r-1} - 1$ cut-edges, which reduces to $\frac{n-7}{3}$ for cubic graphs.*

Proof. Since the contributions not linear in n are negative and we seek an upper bound, the bound holds also for disconnected n-vertex $(2r+1)$-regular graphs. $\square$

Here we use balloons to prove the result of Henning and Yeo [32] minimizing the matching number for n-vertex $(2r+1)$-regular connected graphs; in the next section we characterize the graphs where equality holds.

We use the Berge–Tutte Formula for the matching number. Recall that the deficiency $\mathrm{def}(S)$ of a vertex set S in G is defined by $\mathrm{def}(S) = o(G-S) - |S|$. Tutte [57] proved that a graph G has a 1-factor if and only if $\mathrm{def}(S) \le 0$ for all $S \in V(G)$. The equivalent Berge–Tutte Formula (see Berge [6]) states that $\alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2}(n - \mathrm{def}(S))$.

Lemma 2.1.9. *Let G be an n-vertex $(2r+1)$-regular graph, and let S be a subset of $V(G)$. If the number of edges from each odd component of $G-S$ to S is only 1 or is at least $2r+1$, then* $\mathrm{def}(S) \le \frac{2rb(G)}{2r+1}$.

Proof. Let c_1 be the number of odd components of $G-S$ having one edge to S. By Lemma 2.1.6(a), each component of $G-S$ having one edge to S contains a balloon. Thus $c_1 \le b(G)$. Counting the edges joining S to odd components of $G - S$ yields

$$(2r+1)|S| \geq (2r+1)o(G-S) - 2rc_1 \geq (2r+1)o(G-S) - 2rb(G),$$

and hence $\mathrm{def}(S) = o(G-S) - |S| \leq \frac{2rb(G)}{2r+1}$. □

Corollary 2.1.10. *If G is a connected cubic graph, then $\alpha'(G) \geq \frac{n}{2} - \left\lfloor \frac{b(G)}{3} \right\rfloor$.*

Proof. In a 3-regular graph, all edge-cuts between sets of odd size have odd size, which is 1 or at least 3. Hence Lemma 2.1.9 yields the claim (using the floor function in the second term is valid because $\alpha'(G)$ and $n/2$ are integers). □

If in a connected graph G some set of maximum deficiency satisfies the hypothesis of Lemma 2.1.9, then $\alpha'(G) \geq \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$, by the Berge–Tutte Formula and Lemma 2.1.4. We prove this bound for all connected odd-regular graphs and determine the extremal graphs.

Theorem 2.1.11. *If $G \in \mathcal{F}_{n,r}$, then $\alpha'(G) \geq \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$, with equality if and only if $G \in \mathcal{H}_r$.*

Proof. By the Berge–Tutte Formula, it suffices to show that every set $S \subseteq V(G)$ has deficiency at most $r\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$. By Lemma 2.1.9, we may assume that there is an odd component of $G - S$ such that the number of edges from $G - S$ to S is between 3 and $2r - 1$; call such a component of $G - S$ a *bad subgraph.*

For each edge e joining S to a bad subgraph, replace e with a cut-edge incident to a copy of B_r at its end outside S. Also delete all vertices in bad subgraphs. Let G' denote the resulting graph; note that G' is $(2r+1)$-regular. Unfortunately, G' may be disconnected.

Let c be the number of bad subgraphs, and let x be the total number of vertices in them. Let y be the total number of edges in G joining S to bad subgraphs; y is the number of balloons added in forming G'.

Let p be the number vertices in some bad subgraph Q. If $p \leq 2r + 1$, then regularity forces each vertex of Q to have at least $2r + 2 - p$ neighbors in S. Hence the number of edges from S to

$V(Q)$ is at least $p(2r+2-p)$, which is at least $2r+1$, contradicting that Q is a bad subgraph. We conclude that $p \geq 2r+3$, and hence $x \geq (2r+3)c$.

The number of vertices in G' is $n - x + (2r+3)y$. We also need the number of components of G'. Each time we pull an edge off a bad subgraph Q and make it incident to a copy of B_r, we increase the number of components by 0 or 1. Doing this with the last edge to Q (and deleting $V(Q)$) does not change the number of components. Since G is connected, we conclude that G' has at most $1 + y - c$ components.

The alteration from G to G' ensures that S satisfies the hypotheses of Lemma 2.1.9 for G'. Lemma 2.1.9 does not require connected graphs, so $\mathrm{def}_{G'}(S) \leq \frac{2rb(G')}{2r+1}$. However, applying Lemma 2.1.4 to replace the number of balloons with upper bounds in terms of the number of vertices does require connected graphs. Therefore, we apply Lemma 2.1.4 to each component of G'. We obtain an additive constant 2 in the numerator for each component. Thus $b(G') \leq \frac{(2r-1)(n-x+(2r+3)y)+2(1+y-c)}{4r^2+4r-2}$. With $x \geq (2r+3)c$, we have $b(G') \leq \frac{(2r-1)n+2}{4r^2+4r-2} + \frac{4r^2+4r-1}{4r^2+4r-2}(y-c)$.

Meanwhile, we must also relate $\mathrm{def}_{G'}(S)$ to $\mathrm{def}_G(S)$. We have replaced c odd components in $G - S$ with y odd components in $G' - S$. Thus

$$\begin{aligned}
\mathrm{def}_G(S) &= \mathrm{def}_{G'}(S) - (y-c) \leq \frac{2rb(G')}{2r+1} - (y-c) \\
&\leq \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1} + \frac{2r}{2r+1}\frac{4r^2+4r-1}{4r^2+4r-2}(y-c) - (y-c)
\end{aligned}$$

Thus it suffices to show that $2r(4r^2+4r-1) \leq (2r+1)(4r^2+4r-2)$. This inequality has the form $ab \leq (a+1)(b-1)$ with $a < b$, and hence it holds. □

Corollary 2.1.12. *If G is a connected n-vertex cubic graph, then $\alpha'(G) \geq \frac{4n-1}{9}$, and this is sharp infinitely often.*

We thank Alexandr Kostochka for pointing out a flaw in our original proof of Theorem 2.1.11.

We proved in Proposition 2.1.2 that equality holds in the bound of Theorem 3.1.7 when $G \in \mathcal{H}_r$. Now we show that these are the only graphs achieving equality. Recall that $\mathcal{T}_r$ is the family of trees from which graphs in $\mathcal{H}_r$ are formed by appending small balloons at leaves.

Lemma 2.1.13. *If T is an n-vertex tree in which every non-leaf vertex has degree $2r+1$, then $\alpha'(T) \geq \frac{n-1}{2r+1}$, with equality only when $T \in \mathcal{T}_r$.*

Proof. Since T has $n-1$ edges and maximum degree $2r+1$, the number of vertices needed to cover $E(T)$ is at least $\frac{n-1}{2r+1}$, and hence the König–Egerváry Theorem yields $\alpha'(T) \geq \frac{n-1}{2r+1}$.

If all leaves lie in the same partite set, then the other partite set is a vertex cover of size $\frac{n-1}{2r+1}$. Conversely, equality holding requires a vertex cover Q of size $\frac{n-1}{2r+1}$. No two vertices of Q can cover the same edge, so Q is an independent set. Also every vertex adjacent to a leaf must be in Q, since a leaf covers only one edge.

To show that all leaves are in the same partite set, let x and y be leaves, and let P be the x,y-path in T. The edges of P must be covered by vertices on P, so Q contains a vertex of each edge of P. Since Q is independent, the vertices of P alternate between Q and not-Q, with the neighbors of x and y being in Q. Hence the distance between x and y is even, and they are in the same partite set. □

For a graph $G \in \mathcal{F}_{n,r}$ that achieves the minimum value of the matching number, we show that $G \in \mathcal{H}_r$ by showing that if we shrink each balloon to a single vertex, then the resulting graph is in $\mathcal{T}_r$.

Theorem 2.1.14. *If $G \in \mathcal{F}_{n,r}$ and $\alpha'(G) = \frac{n}{2} - \frac{r}{2}\frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$, then $G \in \mathcal{H}_r$.*

Proof. Equality in the bound requires equality in all the inequalities of Theorem 3.1.7. A set S with maximum deficiency must satisfy $\mathrm{def}(S) = \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1}$. Since the coefficient on $y-c$ in the final displayed inequality for Theorem 3.1.7 is negative, we must have $y = c$. This states that the total number of edges joining S to bad subgraphs equals the number of bad subgraphs, which implies that one edge goes to each bad subgraph, and therefore they are not bad. We conclude that $y = c = 0$, and the number of edges joining S to each odd component of $G - S$ is 1 or is at least $2r+1$.

Now Lemma 2.1.9 applies and yields $\mathrm{def}(S) \leq \frac{2rb(G)}{2r+1}$. From Lemma 2.1.4, we now have

$$\frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1} \leq \frac{2rb(G)}{2r+1} \leq \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1},$$

so $b(G) = \frac{(2r-1)n+2}{4r^2+4r-2}$. From the proof of Lemma 2.1.4, equality in the bound requires each balloon to have exactly $2r+3$ vertices.

Let G' be the graph obtained from G by shrinking each balloon to a single vertex. Let $n' = |V(G')|$ and $m' = |E(G')|$. Since each balloon has $2r+3$ vertices, we have $n = n' + (2r+2)b(G)$. Substituting this expression for n into the formula $b(G) = \frac{(2r-1)n+2}{4r^2+4r-2}$ and simplifying yields $2rb(G) = (2r-1)n' + 2$.

Contraction does not disconnect, so G' is connected. To show that G' is a tree, we count the edges. By the degree sum formula,

$$2m' = (2r+1)n' - 2rb(G) = (2r+1)n' - (2r-1)n' - 2 = 2n' - 2.$$

Finally, we show $G' \in \mathcal{T}_r$. By Lemma 4.1.5, it suffices to show that G' has a matching of size $\frac{n'-1}{2r+1}$. Note that $\alpha'(G') \geq \alpha'(G) - (r+1)b(G)$, and we are given $\alpha'(G) = \frac{n}{2} - \frac{rb(G)}{2r+1}$. Since $\frac{n}{2} - (r+1)b(G) = \frac{n'}{2}$ and $2rb(G) = (2r-1)n' + 2$, we conclude that $\alpha'(G') \geq \frac{n'-1}{2r+1}$. □

Balloons also help in proving bounds on the total domination number. The results are strongest for cubic graphs. We use a lemma proved by Henning that provides a useful upper bound in nearly regular graphs. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum vertex degrees in a graph G.

Lemma 2.1.15. (Henning's Lemma [28]) *If G is a graph with n vertices and m edges, then $\gamma_t(G) \leq n - \frac{m}{\Delta(G)}$.* □

Lemma 2.1.16. *If B is a balloon with p vertices in a cubic graph G, then $\gamma_t(B) \leq \frac{p-1}{2}$. Furthermore, B has a dominating set of size $(p-1)/2$ that contains the neck of B and a neighbor of every vertex other than the neck.*

Proof. Let v be the neck of B. Recall that v has degree 2 in B, and the other vertices of B have degree 3 in B. By Henning's Lemma, $\gamma_t(B) \leq p - (3p-1)/6 = p/2 + 1/6$. Since p is odd and $\gamma_t(B)$ is an integer, $\gamma_t(B) \leq (p-1)/2$.

Let T be the set consisting of v and its two neighbors in B. The number of edges joining T and $V(B) - T$ is 2 or 4, depending on whether T induces a triangle. Note that $B - T$ has

$p-3$ vertices and at least $[3(p-3)-4]/2$ edges. If $\Delta(B-T)=3$, then Henning's Lemma yields $\gamma_t(B-T) \le (p-3)-(3p-9-4)/6 = (p-3)/2+2/3$. Since p is odd and $\gamma_t(B-T)$ is an integer, $\gamma_t(B) \le (p-3)/2$ in this case, and adding v to a smallest total dominating set of $B-T$ yields the desired set.

In the remaining case, $\Delta(B-T) < 3$. Since deleting T removes at most four edges incident to $V(B)-T$, this case requires $p \le 7$. If $p=7$, then $B-T=C_4$, and T is a total dominating set of size $(p-1)/2$ containing v. If $p=5$, then B is the unique smallest balloon B_1, and v with one of its neighbors forms a total dominating set of size $(p-1)/2$. □

When $|V(B)|=7$, it may happen that B has no total dominating set of size $(p-1)/2$ containing its neck. If the neck induces a triangle with its neighbors, then the remaining four vertices induce five edges, and no total dominating set of size 3 contains the neck. Call this special balloon $\hat{B}$.

In addition to small dominating sets, we also need large matchings in balloons.

Lemma 2.1.17. *Every balloon in a 3-regular graph has a matching that covers every vertex except its neck.*

Proof. Let v be the neck of a balloon B, with $N(v)=\{u,w\}$. Let B' consist of two disjoint copies of B plus a cut-edge joining their necks. Now B' is a 3-regular graph with one cut-edge, since B has no cut-edge.

Petersen proved that a 3-regular graphs with at most two cut-edges has a perfect matching. Since B' has odd order, the cut-edge lies in every perfect matching. Deleting it leaves the desired matching in B. □

Since $\alpha'(G) \ge \frac{n}{2} - \frac{b(G)}{3}$ when G is 3-regular and connected (Corollary 2.1.10, proving $\gamma_t(G) \le \frac{n}{2} - \frac{b(G)}{2}$ would yield $\gamma_t(G) \le \alpha'(G)$, with equality only when $b(G)=0$. However, the desired upper bound may fail when G consists of three balloons plus one common neighbor.

The 2-edge-connected case (no balloons) has been well-studied. By Henning's Lemma, $\gamma_t(G) \le n/2$. Equality may hold when G is 2-edge-connected; such graphs were characterized by Henning, Soleimanfallah, Thomassé, and Yeo [30]. The graphs achieving equality consist of two infinite families and one additional 16-vertex graph. In one family, the graph consists of two even cycles with vertex sets $x_1, \dots, x_{2k}$ and $y_1, \dots, y_{2k}$, plus the edges $x_{2i-1}y_{2i}$ and $x_{2i}y_{2i-1}$ for $1 \le i \le k$. Being 2-edge-connected, these graphs also have perfect matchings, so here $\gamma_t(G) = \alpha'(G)$.

Hence we may confine our attention to graphs having balloons. Our strategy is to assemble a small total dominating set S using $(|V(B)|-1)/2$ vertices in each balloon B and $|V(G')|/2$ vertices in the graph G' obtained by deleting the balloons. This gives the desired size. Vertices having neighbors in balloons have degree less than 3 in G'. Such a vertex in S does not need a neighbor in $S \cap V(G')$; Lemma 2.1.16 allows us to give it the neck of the balloon as a neighbor. This weakened restriction on S as a dominating set in G' motivates the following definition.

Definition 2.1.18. *A dominating set S in a graph G is a* semitotal dominating set *(abbreviated* SD-set*) if every vertex with maximum degree in G has a neighbor in S.*

In an SD-set, vertices of non-maximum degree can dominate themselves. The problem of finding an SD-set, like the problem of finding a total dominating set, can be modeled using hypergraphs. In the generalization of graphs to hypergraphs, any vertex set can form an edge; graphs are 2-uniform hypergraphs.

Definition 2.1.19. *A k-*uniform hypergraph *is a hypergraph in which every edge has size k. The* transversal number *$\tau(H)$ of a hypergraph H is the minimum size of a set of vertices that intersects every edge.*

For any graph, the total domination number equals the transversal number of the hypergraph on the same vertex set in which the edges are the vertex neighborhoods. An SD-set corresponds to a transversal when the edge of the hypergraph corresponding to a vertex v of non-maximum degree is its closed neighborhood (the neighborhood plus v itself). The theorem of Chvátal and McDiarmid on transversal number of k-uniform hypergraphs provides exactly what we need to find a sufficiently small SD-set in the graph obtained by deleting the balloons. (In [30], the Chvátal–McDiarmid result is used to explore the total domination numbers of regular graphs, noting in particular that $\gamma_t(G) \le n/2$ follows immediately for cubic graphs.)

Theorem 2.1.20. (Chvátal and McDiarmid [18]) *If H is a k-uniform hypergraph with n vertices and m edges, then $\tau(H) \le \frac{\lfloor k/2 \rfloor m + n}{\lfloor 3k/2 \rfloor}$.*

We state the next two results for a graph G' because we will apply them when G' is the graph obtained from a 3-regular graph G by deleting the vertices in the balloons.

Corollary 2.1.21. *If G' is an n-vertex graph in which every vertex has degree $2r+1$ or $2r$, then G' has an SD-set of size at most $\frac{(r+1)n}{3r+1}$.*

Proof. Form the hypergraph H with $V(H) = V(G')$ by letting the edges be the open neighborhoods of vertices with degree $2r+1$ and the closed neighborhoods of vertices with degree $2r$. Thus H is a $(2r+1)$-uniform hypergraph with n vertices and n edges. By Theorem 2.1.20, $\tau(H) \leq \frac{(r+1)n}{3r+1}$. Every transversal of H is an SD-set in G'. $\square$

We thank Zoltán Füredi for pointing out the effectiveness of the Chvátal–McDiarmid Theorem in proving Corollary 2.1.21.

Using the plan we described above, Corollary 2.1.21 implies that $\gamma_t(G) \leq \frac{n}{2} - \frac{b(G)}{2}$ when $\Delta(G) = 3$ and no two balloons have a common neighbor. The remaining case will need special attention; here deleting the balloons leaves a vertex of degree 1.

Theorem 2.1.22. *If G' is a connected n-vertex graph with maximum degree at most 3, and $n > 1$, then G' has a dominating set S of size at most $n/2$ such that every vertex of degree 3 has a neighbor in S.*

Proof. When $\Delta(G') < 3$, an ordinary dominating set suffices. Always some dominating set has at most $n/2$ vertices, since the complement of a minimal dominating set is also dominating. Hence we may assume that $\Delta(G') = 3$. The case $\Delta(G') < 3$ includes the basis step for induction on n.

If $\delta(G') \geq 2$, then Corollary 2.1.21 provides the desired SD-set. When G' has a vertex u of degree 1, let v be the neighbor of u. Let $F = G' - \{u, v\}$. If F has no isolated vertex, then we can apply the induction hypothesis to each component of F to obtain a set with the desired properties. Let T be the union of these sets; note that $|T| \leq (n-2)/2$.

If v has degree 2, then F is connected, and $T \cup \{v\}$ is an SD-set in G'.

Suppose that v has degree 3. If v has no neighbor of degree 1 other than u, then F has no isolated vertices. Now $T \cup \{v\}$ is an SD-set in G' if T contains a neighbor of v, while otherwise $T \cup \{u\}$ is an SD-set.

In the remaining case, v has degree 3 and has another neighbor w of degree 1. In this case, let $F = G' - \{u, w\}$, and let T be the set in F guaranteed by the induction hypothesis (F is connected, since we only deleted vertices of degree 1). If $v \in T$, then $T \cup \{u\}$ is an SD-set in G'. Otherwise, T must contain the remaining neighbor of v to dominate v, and now $T \cup \{v\}$ is an SD-set in G'. □

Theorem 2.1.23. *If G is a connected cubic graph with n vertices, then $\gamma_t(G) \le \frac{n}{2} - \frac{b(G)}{2}$ (except that $\gamma_t(G) \le n/2 - 1$ when $b(G) = 3$ and the three balloons have a common neighbor), and this is sharp for all even values of $b(G)$.*

Proof. Let G' be the graph obtained by deleting all vertices in balloons. If $G' = K_1$, then G consists of three balloons and their common neighbor. Lemma 2.1.16 yields a total dominating set in two of the balloons and a dominating set in the third that combine with one vertex of G' to yield $\gamma_t(G) \le n/2 - 1$.

When G' has more than one vertex, we can apply Theorem 2.1.22 to obtain an SD-set S in G'. For each balloon B, let v be the neck. Use Lemma 2.1.16 to add a set S_B of size $|V(B) - 1|/2$. If the neighbor of v in $V(G')$ is in S, then choose S_B to be a set that contains v and contains a neighbor of every vertex in $V(B) - \{v\}$. If the neighbor of v in $V(G')$ is not in S, then simply choose S_B to be a total dominating set of B. After these contributions from all balloons, the size is at most $\frac{n}{2} - \frac{b(G)}{2}$.

If equality holds in the bound, then G' must have no SD-set of size less than $|V(G')|/2$. Let G' be formed from a cycle C_t by adding a pendant edge at each vertex. An SD-set in G' must use one vertex from each set consisting of a vertex of degree 1 and its neighbor.

We construct our example G by adding two 7-vertex balloons adjacent to each vertex of degree 1 in G'. Each such balloon is the special balloon $\hat{B}$ discussed after Lemma 2.1.16. The number of balloons is $2t$. Recall that $\hat{B}$ has no total dominating set of size 3 that contains its neck. Therefore, if a total dominating set in G avoids some vertex u of degree 1 in G', then the balloons adjacent to u contribute at least four vertices each, and the 16-vertex "wedge" containing them, u, and the neighbor of u in G' contributes at least eight vertices. Using u still requires it to contribute seven vertices, including three from each balloon. Thus we can save only 1 for each pair of balloons, and $\gamma_t(G) = \frac{n}{2} - \frac{b(G)}{2}$. □

Corollary 2.1.10 and Theorem 2.1.23 together improve the inequality $\gamma_t(G) \le \alpha'(G)$ for connected cubic graphs.

Corollary 2.1.24. *If G is a connected n-vertex cubic graph, then $\gamma_t(G) \le \alpha'(G) - b(G)/6$, except when $b(G) = 3$ and there is exactly one vertex outside the balloons, in which case still $\gamma_t(G) \le \alpha'(G)$.*

Proof. From the bounds in Corollary 2.1.10 and Theorem 2.1.23, it suffices to consider the exceptional case. Here $b(G) = 3$, and $\gamma_t(G) = n/2 - 1$ is possible. By Lemma 2.1.17, there are matchings in the balloon that cover all but the neck. One of the necks can be matched to their common neighbor, leaving only the two other necks as uncovered vertices. Hence $\alpha'(G) = n/2 - 1$ (equality holds, because deleting the vertex outside the balloons leaves three odd components). $\square$

The 3-regular case is the only case where the inequality between γ_t and α' is delicate. When more edges are added, α' tends to increase and γ_t tends to increase, so the separation increases. For $(2r+1)$-regular graphs, applying the Chvátal–McDiarmid Theorem to the neighborhood hypergraph immediately yields $\gamma_t(G) \le \frac{(r+1)n}{3r+1}$. On the other hand, $\alpha'(G) \ge \left\lfloor \frac{2}{n}\left(1 - \frac{2r-1}{2r+1}\frac{r}{2r^2+2r-1}\right)\right\rfloor$ (Theorem 3.1.7). For large r, this upper bound on $\gamma_t(G)$ tends to $n/3$ and the lower bound on $\alpha'(G)$ tends to $n/2$. Already when $r = 2$, we have $\gamma_t(G) \le 3n/7 < 9n/22 < \alpha'(G)$. Hence the separation between γ_t and α' is already in the coefficient of the linear term, regardless of the number of balloons, and the balloons become important only for the 3-regular case.

Furthermore, the upper bound from the Chvátal–McDiarmid Theorem is not sharp for larger degree. The best-possible upper bounds on $\gamma_t(G)$ when G is k-regular and has n vertices are not known. Yeo [60] conjectured that if G is a connected n-vertex graph with $\delta(G) \ge 4$ other than the bipartite complement of the Heawood graph, then $\gamma_t(G) \le \frac{2}{5}n$.

2.2 Matching and Edge-connectivity

A long time ago, Petersen [53] proved that if a cubic graph has no cut-edges, then it has a perfect matching. It is natural to ask what happens when there are cut-edges. The *matching number* of a graph G is the maximum size of a matching in G. Biedl et al. [8] determined the smallest matching number among connected cubic graphs with n vertices. Henning and Yeo [32] extended this to connected k-regular n-vertex graphs for appropriate n. For k odd, in Section 2.1 and [49] we gave a short proof of their bound, characterized the extremal graphs, and studied the relationship between the matching number and the number of cut-edges. Chartrand et al. [16] determined the minimum number of vertices in a k-regular $(k-2)$-edge-connected graph with no perfect matching. Niessen and Randerath [45] extended this to k-regular l-edge-connected graphs. In another direction, Broere et al. [7] gave a formula for the minimum size of a matching among k-regular $(k-2)$-edge-connected graphs with a fixed number of vertices. Katerinis [34] considered the analogue for vertex connectivity. Our lower bound for the minimum size of a matching in a k-regular l-edge-connected graph with n vertices implies these various results when the parameters are set to appropriate values. Although the bound is sharp infinitely often when $l > 0$, a stronger bound appears in Section 2.1 and [32, 49] for $l = 0$. In Section 3, we characterize the graphs achieving equality.

Since the degree sum of any graph is even, it follows that every edge cut in a regular graph of even degree has even size, and every edge cut in a regular graph of odd degree that breaks the vertex set into odd-sized sets has odd size. Therefore, it suffices to study $(2t+1)$-edge-connected $(2r+1)$-regular graphs and $2t$-edge-connected $2r$-regular graphs. We will show in the next section that the bound is sharp infinitely often and characterize when equality holds, except for $t = 0$ in $(2r+1)$-regular graphs. In that case, the bound of Theorem 3.1.1 can be improved. (See Section 2.1 and [32, 49]).

Since $2r^2 + r = 2(r + \frac{1}{4})^2 - \frac{1}{8}$, the formula in Theorem 2.2.2 has a very similar flavor to that in Theorem 2.2.1. In the special case $t = r-1$, the formulas in Theorem 2.2.1 and Theorem 2.2.2 reduce to essentially the formula in Broere et al. [7]. Also when n is even and less than $2(k\lceil k/2\rceil + k - 1)$, those formulas imply that a $(k-2)$-edge-connected k-regular graph with n vertices has a perfect matching; this is the result of Chartrand et al. [16]. More generally, for l-edge-connected graphs, the threshold on the number of vertices for graphs without perfect matchings in Niessen and Randerath [45] also follows.

We use the Berge–Tutte Formula for the matching number. The deficiency $\mathrm{def}(S)$ of a vertex set S in G is defined by $\mathrm{def}(S) = o(G-S) - |S|$, where $o(H)$ is the number of odd components in a graph H. Tutte [57] proved that a graph G has a 1-factor if and only if $\mathrm{def}(S) \le 0$ for all $S \in V(G)$. The equivalent Berge–Tutte Formula (see Berge [6]) states that $\alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2}(n - \mathrm{def}(S))$.

Theorem 2.2.1. *If G is a $(2t+1)$-edge-connected $(2r+1)$-regular graph with n vertices, where $0 \le t \le r$, then $\alpha'(G) \ge \frac{n}{2} - (\frac{r-t}{2(r+1)^2+t})\frac{n}{2}$.*

Proof. Let S be a set with maximum deficiency. Thus, $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(S))$, where $\mathrm{def}(S) = o(G-S) - |S|$. Let c_i count the odd components of $G - S$ having exactly i edges to S; note that c_i is nonzero only when i is odd. Let $c = c_{(2t+1)} + \cdots + c_{(2r-1)}$, and let $c' = o(G-S) - c$. Each odd component counted by c' has at least $2r+1$ edges to S. Note that for $2t+1 \le i \le 2r-1$, each odd component of $G - S$ having exactly i edges to S has at least $2r+3$ vertices. (If C has q vertices, then $[C, \overline{C}] \ge q(2r+2-q) \ge 2r+1$.) Since the edges incident to S include the edges joining S to odd components of $G - S$, we have $(2r+1)|S| \ge (2r+1)c' + (2t+1)c$, and hence $|S| \ge c' + (\frac{2t+1}{2r+1})c \ge (\frac{2t+1}{2r+1})c$. Therefore, $n \ge |S| + c(2r+3) \ge (\frac{2t+1}{2r+1} + 2r+3)c$, which implies that $c \le (\frac{2r+1}{4r^2+8r+4+2t})n$. Now, we compute

$$\begin{aligned} \mathrm{def}(S) = (c + c') - |S| &\le c - \frac{2t+1}{2r+1}c = \frac{2(r-t)}{2r+1}c \\ &\le \frac{2(r-t)}{2r+1}\left(\frac{2r+1}{4r^2+4r+4+2t}\right)n = \frac{(r-t)n}{2(r+1)^2+t}. \end{aligned}$$

□

As noted earlier, the same bound holds for $2t$-edge-connected $(2r+1)$-regular graphs.

Similarly, the bound in the next theorem also holds for $(2t-1)$-edge-connected $2r$-regular graphs (since every edge cut has even size, every $(2t-1)$-edge-connected $2r$-regular graph $2t$-edge-connected).

Theorem 2.2.2. *If G is a $2t$-edge-connected $2r$-regular graph with n vertices, where $1 \le t \le r$ and $r \ge 2$, then $\alpha'(G) \ge \frac{n}{2} - (\frac{r-t}{2r^2+r+t})\frac{n}{2}$.*

Proof. The proof is similar to that of Theorem 2.2.1. Defining S and c_i as in that proof, here we have that i is even and at least $2t$. Also, for $2t \le i \le 2r-2$, the odd component of $G - S$ having i edges to S has at least $2r+1$ vertices. The same steps as before then lead to $\mathrm{def}(S) \le \frac{(r-t)n}{2r^2+r+t}$. □

We begin by developing properties that graphs achieving equality in the bounds of Theorem 2.2.1 and Theorem 2.2.2 must satisfy. We will show that all graphs with these properties meet the bound, thereby characterizing equality. Finally, we give an explicit construction of infinitely many graphs achieving equality, for each fixed r and t with $r > t > 0$. The needed properties lead us to define special families.

Definition 2.2.3. A *nontrivial cut* in a graph G is an edge cut with at least two vertices on each side. A $(2r+1, 2t+1)$-*bullet* is a graph H satisfying the following conditions :

(1) $|V(H)| = 2r + 3$,

(2) $\delta(H) \geq \max\{2r - 2t, r + 1\}$,

(3) $\Delta(H) = 2r + 1$,

(4) $|E(\overline{H})| = r + t + 2$ and

(5) every nontrivial cut has at least $2r + 1$ edges.

Definition 2.2.4. Let B be a graph with $\Delta(B) = a$ such that $\sum_{v \in V(B)}(a - d_B(v)) = b$. If u is a vertex of degree b in a graph H, then *splicing* B *into* u means deleting u and replacing each edge of the form uw in it with an edge from w to a vertex of B, in such a way that each vertex of B now has degree a.

Definition 2.2.5. An (a, b)-*biregular* graph is a bipartite graph with partite sets A and B such that vertices in A have degree a, and those in B have degree b. Let $\mathcal{T}_{r,t}$ be the family of $(2t+1)$-edge-connected $(2r+1, 2t+1)$-biregular graph, let $\mathcal{B}_{r,t}$ be the family of $(2r+1, 2t+1)$-bullets, and let $\mathcal{H}_{r,t}$ be the family of graphs obtained from a graph H in $\mathcal{T}_{r,t}$ by splicing a $(2r+1, 2t+1)$-bullet into each vertex having degree $2t + 1$ in H.

Figure 2.4 describes a bullet $B_{2,1}$, a $(5, 3)$-biregular graph, and how to splice $B_{2,1}$ into a $(5, 3)$-biregular graph.

Lemma 2.2.6. *Every graph in $\mathcal{H}_{r,t}$ is $(2t + 1)$-edge-connected and $(2r + 1)$-regular graph.*

Proof. Let H be a graph in $\mathcal{T}_{r,t}$ with partite sets R and T such that vertices in R have degree $2r+1$ and those in T have $2t + 1$. If $G \in \mathcal{H}_{r,t}$ is derived from H by splicing bullets into vertices of T,

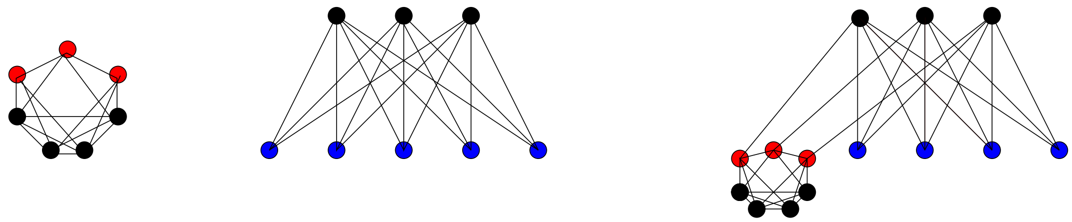

Figure 2.4: $B_{2,1}$, a $(5,3)$-biregular graph, and splicing

then by the construction, every vertex in G has degree $2r+1$. For the other property, it suffices to show that splicing a bullet B into one vertex u of degree $2t+1$ in a $(2t+1)$-edge-connected graph K yields a $(2t+1)$-edge-connected graph K'. Let F be a set of edges in K' with $|F| \leq 2t$. Since K is $(2t+1)$-edge-connected, $G-F$ is connected, where edges joining $V(K'-u)$ to $V(B)$ correspond to edges joining $V(K-u)$ to u. Hence $K'-F$ has a path from each vertex of $V(K')-V(B)$ to $V(B)$. Hence it suffices to show that each vertex of B can reach every other vertex of B in $K'-F$. If $V(B)$ does not induce a connected subgraph in $K'-F$, then F contains an edge cut of B. Since B is a bullet, every nontrivial edge cut has size at least $2r+1$. Hence F contains all edges of B incident to some vertex x. Let $S = V(B) - x$. In $K'-F$, the subgraph induced by S is connected, since otherwise F contains a nontrivial edge cut of B. Hence it suffices to show that $K'-F$ has a path from x to S through vertices outside B. Since $d_{K'}(x) = 2r+1 > 2t$ and F contains all edges of B incident to x, some edge from x to $V(K')-V(B)$ remains in $K'-F$; let y be a neighbor of x via such an edge. Also, since $d_B(x) \geq r+1 > t+1$, there are fewer edges from x to $V(K')-V(B)$ than to S. Hence $|[S,\overline{S}]| \geq 2t+1$, and an edge e remains in $K'-F$ from S to $V(K')-V(B)$; let w be the endpoint outside S. Since F has at least one edge in B, $K-F$ is 2-edge-connected. Hence it has a cycle C through uy and uw. Now $C-u$ completes a path with xy and e from x to S in $K'-F$. □

Theorem 2.2.7. *For $t, r \in \mathbb{N}$ with $t < r$, a $(2t+1)$-edge-connected $(2r+1)$-regular graph G achieves equality in the bound of Theorem 2.2.1 if and only if it is in $\mathcal{H}_{r,t}$.*

Proof. First, suppose that G is a graph in $\mathcal{F}_{r,t}$ derived from a $(2t+1)$-edge-connected $(2r+1, 2t+1)$-biregular graph H in $\mathcal{H}_{r,t}$ by splicing in bullets. By Lemma 2.2.6, G is $(2t+1)$-edge-connected and

$(2r+1)$-regular. Let R and T be the sets of vertices with degree $2r+1$ and degree $2t+1$ in H, respectively. Note that $|T| = \frac{2r+1}{2t+1}|R|$ and $|V(G)| = |R| + (2r+3)|T|$. Hence $|R| = \frac{(2t+1)n}{4r^2+8r+r+2t}$. Also, there are $|T|$ odd components in $G - R$, which implies that $\mathrm{def}(G) \leq \mathrm{def}(R) = |T| - |R| = (\frac{2r+1}{2t+1} - 1)|R| = \frac{2(r-t)}{2t+1}\frac{2t+1}{(2r+1)(2r+3)+2t+1} = \frac{(r-t)n}{2(r+1)^2+t}$. Theorem 2.2.1 yields $\mathrm{def}(G) \leq \frac{(r-t)n}{2(r+1)^2+t}$; hence equality holds.

Conversely, we want to show that every graph G achieving equality in Theorem 2.2.1 is in $\mathcal{H}_{r,t}$. By definition, G is $(2t+1)$-edge-connected and $(2r+1)$-regular. Let S be a maximal vertex subset with maximum deficiency in G. By this maximality, $G - S$ has no even components. Achieving equality in the computation of Theorem 2.2.1 requires the following conditions :

(i) for $i \geq 2t+3$, no odd component in $G - S$ has i edges to S,

(ii) every odd component of $G - S$ has exactly $2r+3$ vertices, and

(iii) S is an independent set.

Thus, if H is the graph obtained from G by shrinking each odd component in $G - S$ to a single vertex, then the resulting graph is in $\mathcal{H}_{r,t}$, and G is obtained by splicing each odd component of $G - S$ into a vertex of H with degree $2t+1$. (Note that if H is not $(2t+1)$-edge-connected, then G would not be.)

Now, we consider an odd component C of $G-S$. It remains only to show that C is a $(2r+1, 2t+1)$-bullet. Suppose that A is a proper nonempty vertex subset of C, and let $l = |[A, \overline{A}]|$, where $\overline{A} = V(C) - A$. We may assume that $|A| \leq |\overline{A}|$. Letting $a = |A|$, we then have $a \leq r+1$. We show that $l \geq 2r+1$, except possibly when $a = 1$. We have $(2r+1)a = \sum_{v \in A} d_G(v) \leq a(a-1) + l + 2t + 1$, which implies that

$$a(2r+2-a) \leq l + 2t + 1 \quad (1).$$

When $2 \leq a \leq r+1$, always $a(2r+2-a) \geq 4r$. Thus, $l \geq 4r - 2t - 1 \geq 4r - 2(r-1) - 1 = 2r+1$.

If $a = 1$, then (1) yields $l \geq 2r - 2t$. Let $b = |[A, S]|$ and $c = |[\overline{A}, S]|$. Note that $b + c = 2t+1$ and $l + b = 2r+1$, which yields $l - c = 2r - 2t$. Since G is $(2t+1)$-edge-connected, $l + c \geq 2t+1$. Adding $l - c = 2r - 2t$ yields $2l \geq 2r+1$, which implies that $l \geq r+1$. Thus, $\delta(C) \geq \max\{2r-2t, r+2\}$.

Since G is $(2r+1)$-regular and vertices of C together lose only $2t+1$ incident edges, it follows that $\Delta(C) = 2r+1$. Since C has $\frac{[(2r+1)(2r+3)-(2t+1)]}{2}$ edges, we have $|E(\overline{C})| = r+t+2$. Finally, we have shown that nontrivial cuts in C have size at least $2r+1$. Hence $C \in \mathcal{B}_{r,t}$. $\square$

Similarly, we can characterize when the matching number for even-regular graphs is minimized. When the parameters are even, we use a slightly different definition of bullet.

Definition 2.2.8. A $(2r, 2t)$-*bullet* is a graph H satisfying the following conditions :

(1) $|V(H)| = 2r+1$,

(2) $\delta(H) \geq \max\{2r-2t, r\}$,

(3) $\Delta(H) = 2r$,

(4) $|E(\overline{H})| = t$ and

(5) every nontrivial cut has at least $2r$ edges.

Let $\mathcal{T}'_{r,t}$ be the family of $2t$-edge-connected $(2r, 2t)$-biregular bipartite graphs, let $\mathcal{B}'_{r,t}$ be the family of $(2r, 2t)$-bullets, and let $\mathcal{H}'_{r,t}$ be the family of graphs obtained from a graph H in $\mathcal{T}'_{r,t}$ by splicing a $(2r, 2t)$-bullet in $\mathcal{B}'_{r,t}$ into each vertex having degree $2t$ in H'.

Arguments similar to the proofs of Lemma 2.2.6 and Theorem 2.2.7 yield the following results.

Lemma 2.2.9. *Every graph in $\mathcal{H}'_{r,t}$ is $2t$-edge-connected and $2r$-regular graph.*

Theorem 2.2.10. *For $t, r \in \mathbb{N}$ with $t < r$, a $2t$-edge-connected $2r$-regular graph G achieves equality in the bound of Theorem 3.1.1 if and only if it is in $\mathcal{H}'_{r,t}$.*

Finally, we show that there are infinitely many graphs in the families $\mathcal{H}_{r,t}$ and $\mathcal{H}'_{r,t}$. It suffices to have at least one $(2r+1, 2t+1)$-bullet and $(2r, 2t)$-bullet and infinitely many graphs in $\mathcal{T}_{r,t}$ and $\mathcal{T}'_{r,t}$. Let $B_r = \overline{P_3 + rK_2}$. For $0 \leq t \leq r$, let $B_{r,t}$ and $B'_{r,t}$ be graphs obtained from B_r and K_{2r+1} respectively, by deleting a matching of size t whose elements are not incident to degree $2r$ and $2r-1$, respectively. We show first that $B_{r,t} \in \mathcal{B}_{r,t}$ and $B'_{r,t} \in \mathcal{B}'_{r,t}$.

To determine the edge-connectivity of $B_{r,t}$ and $B'_{r,t}$, we use the following standard exercies.

Lemma 2.2.11. *If an n-vertex graph G is connected, and $\frac{n}{2} \leq \delta(G) \leq n$, then $\kappa'(G) = \delta(G)$.*

Proof. If $[S, \overline{S}]$ with $|S| \leq \frac{n}{2}$ is an edge-cut in G, then $|[S, \overline{S}]| \geq |S|(\delta(G) + 1 - |S|) \geq \delta(G)$. $\square$

Corollary 2.2.12. *If $0 \leq t \leq r$, then $\kappa'(B_{r,t}) = 2r$ and $\kappa'(B'_{r,t}) = 2r-1$.*

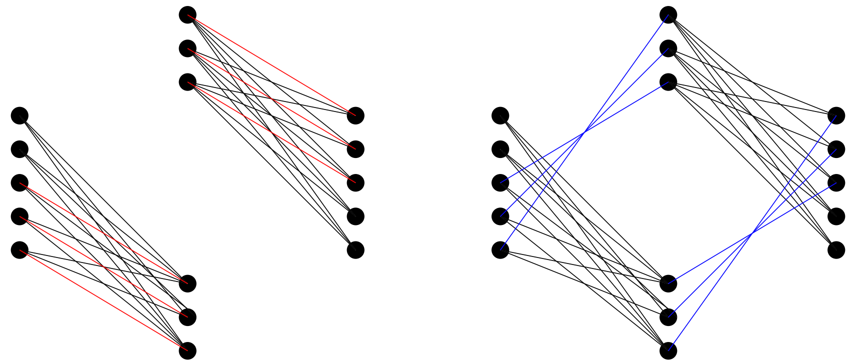

Figure 2.5: Two copies of $K_{5,3}$ and G_2 constructed from two copies of $K_{5,3}$

Thus, $B_{r,t} \in \mathcal{B}_{r,t}$ and $B'_{r,t} \in \mathcal{B}'_{r,t}$.

When H is a (a, b)-biregular graph with edge-connectivity b, where $a = 2r + 1$ and $b = 2t + 1$, splicing $B_{r,t}$ into each vertex having degree $2t + 1$ in H preserves $(2t + 1)$-edge-connectedness by Lemma 2.2.6. Similarly, for $a = 2r$ and $b = 2t$, splicing $B'_{r,t}$ into each vertex having degree $2t$ in H preserves $2t$-edge-connectedness by Lemma 2.2.9.

By the above statements, to show that there are infinitely many graphs in the families $\mathcal{F}_{r,t}$ and $\mathcal{F}'_{r,t}$, it suffices to show that there are infinitely many (a, b)-biregular graphs with edge-connectivity b.

Example 2.2.13. *Construction of G_k First, put k copies of $K_{a,b}$ around a circle and modify them to construct G_k as follows:*

For $1 \leq i \leq k$, and $a \geq b$, let H_i be a copy of $K_{a,b}$ with partite sets X_i and Y_i of sizes b and a, respectively. Choose $S_i \subseteq X_i$ and $T_i \subseteq Y_i$, both of size b. Delete a matching of size b joining S_i and T_i. Restore the original vertex degrees by adding a matching of size b joining T_i and S_{i+1} for each i, with subscript taken modulo k. The resulting graph is G_k.

Figure 2.5 describes how to construct G_2 from two copies of $K_{5,3}$.

Elementary lemmas lead us to $\kappa'(G_k) = b$.

Lemma 2.2.14. *[54] If G is bipartite with diameter at most 3, then $\kappa'(G) = \delta(G)$.*

Since for $a \geq b$, the complete bipartite graph $K_{a,b}$ has diameter 2, by the above lemma, $\kappa'(K_{a,b}) = b$.

Corollary 2.2.15. *For $a \geq b$, if a graph H is a graph obtained from $K_{a,b}$ by deleting a matching of size b, then $\kappa'(H) = b - 1$.*

Proof. Since the graph H has diameter 3, by the above lemma, $\kappa'(H) = b - 1$. $\square$

Theorem 2.2.16. *The edge-connectivity of G_k is equal to b.*

Proof. Since there is a vertex with degree b, $\kappa'(G_k) \leq b$. To prove $\kappa'(G_k) = b$, consider $F \subseteq E(G_k)$ with $|F| < b$. Let $G' = G_k - F$; we show that G' is connected. By Corollary 2.2.15, each H_i' in the construction of Example 2.2.13 is $(b-1)$-edge-connected. Therefore, the subgraph of G' induced by $S_i \cup T_i$ is connected unless $F \subseteq E(H_i')$. Also, since $|F| < b$, there exists an edge in G' joining T_i and S_{i+1}. Either what remains of each H_i' is connected and the subgraphs are each linked to the next, or one H_i is cut but all edges outside remain. In this special case, every vertex except $V(H_i)$ is connected each other. Now, we need to check on the vertex $V(H_i)$. If we delete edges less than b in H_i, then some vertices in S_i are not incident to T_i, and the other vertices in S_i are still incident to H_i. The vertices in S_i, which are not incident to T_i have paths to the other vertices through S_{i-1}, and the vertices in S_i, which are incident to T_i have paths to the other vertices through T_i and S_{i+1}. Finally, G' is connected.

$\square$

Corollary 2.2.17. *There are infinitely many graphs in $\mathcal{H}_{r,t}$ and $\mathcal{H}'_{r,t}$.*

Proof. Use Theorem 2.2.16 with $a = 2r + 1$ and $b = 2t + 1$. $\square$

Chapter 3

Edge-connectivity, Matching, and Eigenvalues in Regular Graphs

A lot of research in graph theory over the last 40 years was stimulated by a classical result of Fiedler [25], stating that $\kappa(G) \geq \mu_2(G)$ for a non-complete graph G, where $\kappa(G)$ is the connectivity of G and $\mu_2(G)$ is the second smallest eigenvalue of the Laplacian matrix.

In Section 3.1, we study the relationship between eigenvalues and the existence of certain subgraphs in regular graphs. We give a condition on an appropriate eigenvalue that guarantees a lower bound for the matching number of a t-edge-connected d-regular graph, when $t \leq d-2$. This work extends some classical results of von Baebler and Berge and more recent work of Cioabă, Gregory, and Haemers. We also study the relationships between the eigenvalues of a d-regular t-edge-connected graph G and the maximum number of pairwise disjoint connected subgraphs in G that are each joined to the rest of the graph by exactly t edges.

In Section 3.2, we study the problem of finding the weakest hypothesis on the second largest eigenvalue λ_2 for a d-regular graph G to guarantee that G is l-edge-connected.

3.1 Edge-connectivity, Matching, and Eigenvalues

The eigenvalues of a d-regular graph G are closely related to many important properties of G (see [10, 27, 36]). In particular, the second largest eigenvalue of G is closely related to the edge-distribution of G. When S and T are subsets of vertices of G, denote by $[S,T]$ the set of edges with one endpoint in S and one endpoint in T and let $e(S,T) = |[S,T]|$. If G is a d-regular n-vertex graph with second largest eigenvalue λ_2 and S a nonempty proper subset of $V(G)$, then

$$e(S, V(G) \setminus S) \geq \frac{(d-\lambda_2)|S|(n-|S|)}{n}.$$

A set M of edges of a graph G is a *matching* if each vertex of G is contained in at most one edge of M. The *matching number* $\alpha'(G)$ is the maximum size of a matching in G. A graph is *t-edge-connected* if the removal of any $t-1$ edges does not disconnect it. The *eigenvalues* of G are the eigenvalues of its adjacency matrix. The *adjacency matrix* of G has its rows and columns indexed by the vertices of G, and the (i,j)-th entry of A is 1 if i and j are adjacent and 0 otherwise. If G has n vertices, then let $\lambda_1(G), \ldots, \lambda_n(G)$ be its eigenvalues indexed in nonincreasing order. It is well known that if G is a connected d-regular graph, then $\lambda_1 = d > \lambda_2$ and $\lambda_n \geq -d$, with equality if and only if G is bipartite.

In this section, we show that many other eigenvalues of a regular graph G are related to the existence of various substructures in G. A well-known result in graph theory due to von Baebler [4] (for d odd) and Berge [5] states that any d-regular $(d-1)$-edge-connected graph contains a perfect matching. This extends the work of Petersen [53], who showed that a 3-regular graph without cut-edges contains a perfect matching. Recently, Cioabă, Gregory, and Haemers [15] found the best possible conditions on the eigenvalues of a d-regular graph that guarantee the existence of a perfect matching. Their work improved previous results of various authors (see [9, 12, 14, 36]). In this section, we determine connections between the eigenvalues of a t-edge-connected d-regular graph and its matching number when $t \leq d-2$. Our work can be seen both as an extension of the work of von Baebler and Berge (for lower values of the edge-connectivity) and as an extension of the results of Cioabă, Gregory, and Haemers (for higher values of the edge-connectivity).

Our main result in this direction is the following theorem.

Theorem 3.1.1. *Denote by θ the greatest solution of the equation $x^3 - x^2 - 6x + 2 = 0$, and let*

$$\rho(d) = \begin{cases} \theta & \text{if } d = 3 \\ \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \geq 4 \text{ is even} \\ \frac{d-3+\sqrt{(d+1)^2+16}}{2} & \text{if } d \geq 5 \text{ is odd.} \end{cases} \tag{3.1}$$

Let $p \geq 3$ be an integer. If G is a t-edge-connected d-regular graph such that $\lambda_p(G) < \rho(d)$, then

$$\alpha'(G) > \begin{cases} \frac{n-p+\lfloor \frac{tp}{d} \rfloor}{2} & \text{when } d \equiv t \pmod 2 \\ \frac{n-p+\lfloor \frac{(t+1)p}{d} \rfloor}{2} & \text{when } d \equiv t+1 \pmod 2. \end{cases}$$

We present examples that show that our result is best possible when $t = d-2$; for each d and for infinitely many values of p, we construct d-regular graph H with edge-connectivity $d-2$ having $\lambda_p(H) = \rho(d)$ and $\alpha'(H) = \frac{n-p+\lfloor \frac{tp}{d} \rfloor}{2}$.

Henning and Yeo [32] determined the minimum value of the matching number of a connected d-regular graph of order n. In Chapter 2 ([49] [50]), we extended their results and obtained a relationship between the matching number of a connected d-regular graph G and the number of balloons in G. A balloon is a maximal 2-edge-connected subgraph that is joined to the rest of G by exactly one cut-edge. Let $b(G)$ denote the number of balloons of G. Obviously $b(G) = 0$ when d is even as G contains no cut-edges in this case. If b denotes the maximum possible number of balloons in a d-regular graph with n vertices, we proved $\alpha'(G) \geq \frac{n}{2} - \frac{(d-1)b}{2d}$ when d is odd, and they showed that this inequality implies the result of Henning and Yeo from [32].

The number of balloons is also related to other combinatorial invariants such as the total domination number of G (see Section 2.2 and [49]) and the number of cut-edges in G. If $c(G)$ denotes the number of cut-edges of G, then $b(G) \leq c(G)$ when $c(G) \geq 2$. Motivated by the connections between balloons and these combinatorial invariants, we study the relationship between the number of balloons in G and its eigenvalues. Our result is the following theorem.

Theorem 3.1.2. *When d is an odd integer with $d \geq 3$, let $\theta(d)$ denote the largest solution of the equation*

$$x^3 - (d-2)x^2 - 2dx + d - 1 = 0.$$

If k is an integer with $k \geq 3$ and G is a connected d-regular graph such that $\lambda_k(G) < \theta(d)$, then $b(G) \leq k-1$.

We also show for each d that this result is best possible for infinitely many values of k by presenting examples of d-regular graph H having $\lambda_k(H) = \theta(d)$ and $b(G) = k$.

Note that for $k = 2$, the following result was proved in [12].

Theorem 3.1.3 ([12]). *Let d be an odd integer with $d \geq 3$ and denote by $\gamma(d)$ the largest root of the equation*

$$x^3 - (d-3)x^2 - (3d-2)x - 2 = 0.$$

If G is a d-regular graph with $\lambda_2(G) < \gamma(d)$, then $b(G) = 0$ (equivalently, G is 2-edge-connected).

Many authors have studied the number of cut-edges or, more generally, the number of smallest edge-cuts of a graph (see [20, 31, 38, 44]). In this paper, we determine a relationship between eigenvalues and a parameter closely related to the number of smallest edge-cuts. If G is a t-edge-connected graph, then let $b_t(G)$ denote the maximum number of pairwise disjoint connected subgraphs $H_1, \ldots, H_l$ of G with the property that $e(V(H_i), \overline{V(H_i)}) = t$ for $1 \leq i \leq l$. If $c_t(G)$ denotes the number of edge-cuts of size t of G, then $b_t(G) \leq c_t(G)$ when $c_t(G) \geq 2$. Our main result in this direction is the following theorem.

Theorem 3.1.4. *Let d and t be two integers of the same parity with $d > t \geq 1$. If $p \geq 3$ is an integer and G is a t-edge-connected d-regular graph with*

$$\lambda_p(G) < \begin{cases} \frac{d-2+\sqrt{(d+2)^2-4t}}{2} & \text{if } d \text{ and } t \text{ are even} \\ \frac{d-3+\sqrt{(d+3)^2-4t}}{2} & \text{if } d \text{ and } t \text{ are odd} \end{cases} \tag{3.2}$$

then $b_t(G) \leq p-1$.

We also show for each d that this result is best possible for infinitely many values of p by presenting examples of d-regular t-edge-connected graphs H having $\lambda_p(H)$ equal to the right-hand side of (3.2) and $b_t(G) = p$.

We remark that for $t = 2$ and $p = 2$, the following result was proved in [12].

Theorem 3.1.5 ([12]). *If G is a d-regular graph with $\lambda_2(G) < \frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then $b_2(G) = 0$ (or equivalently, G is 3-edge-connected).*

The main tool in our arguments is eigenvalue interlacing (see [10, 27, 33]). Let $\lambda_j(M)$ be the j-th largest eigenvalue of a matrix M.

Lemma 3.1.6. *(***Interlacing Theorem***) If A is a real symmetric $n \times n$ matrix and B is a principal submatrix of A with order $m \times m$, then for $1 \leq i \leq m$,*

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A).$$

This theorem implies that if H is an induced subgraph of a graph G, then the eigenvalues of H interlace the eigenvalues of G. Consider a partition $V(G) = V_1 \cup \cdots \cup V_s$ of the vertex set of G into s non-empty subsets. For $1 \leq i, j \leq s$, let $b_{i,j}$ denote the average number of neighbours in V_j of the vertices in V_i. The quotient matrix of this partition is the $s \times s$ matrix whose (i, j)-th entry equals $b_{i,j}$. The eigenvalues of the quotient matrix interlace the eigenvalues of G. This partition is *equitable* if for each $1 \leq i, j \leq s$, any vertex $v \in V_i$ has exactly $b_{i,j}$ neighbours in V_j. In this case, the eigenvalues of the quotient matrix are eigenvalues of G and the spectral radius of the quotient matrix equals the spectral radius of G (see [10, 27] for more details).

The deficiency $\text{def}(S)$ of a vertex set S in G is defined by $\text{def}(S) = o(G-S)-|S|$, where $o(H)$ is the number of components of H having an odd number of vertices. Tutte [57] proved that a graph G has a perfect matching if and only if $\text{def}(S) \leq 0$ for all $S \in V(G)$. The equivalent Berge-Tutte Formula (see Berge [6] and also [39]) states that

$$\alpha'(G) = \frac{1}{2}(n - \max_{S \subseteq V} \text{def}(S)). \tag{3.3}$$

Lemma 3.1.7. *Let G be a t-edge-connected d-regular graph with n vertices and let r be an integer with $r \geq 2$. If $\alpha'(G) \leq \frac{n-r}{2}$, then*

$$\rho(d) \leq \begin{cases} \lambda_{\lceil \frac{rd}{d-t} \rceil} & \text{if } d \equiv t \pmod 2, \\ \lambda_{\lceil \frac{rd}{d-(t+1)} \rceil} & \text{if } d \equiv t+1 \pmod 2. \end{cases} \tag{3.4}$$

Proof. Let S be a subset of G with maximum deficiency. Let $\mathcal{O}_1, \ldots, \mathcal{O}_q$ be the odd components of $G - S$. Because $\alpha'(G)$ is at most $\frac{n-r}{2}$, we have $q \geq |S| + r$. Let $n_i = |V(\mathcal{O}_i)|, e_i = |E(\mathcal{O}_i)|$ and $t_i = e(S, \mathcal{O}_i)$ for $1 \leq i \leq q$. By the degree-sum formula, d and t_i have the same parity. Because G

is t-edge-connected, we have $t_i \geq t$ when $d \equiv t \pmod 2$ and $t_i \geq t+1$ when $d \equiv t+1 \pmod 2$.

We will prove the lemma for $d \equiv t \pmod 2$. The proof of the other case is similar and will be omitted.

By counting the edges between S and $V(G) \setminus S$, we have $d|S| \geq e(S, V(G) \setminus S) \geq \sum_{i=1}^{q} t_i \geq qt \geq (|S|+r)t$. Thus, $(d-t)|S| \geq rt$, which implies that $|S| \geq \frac{rt}{d-t}$. Because $q \geq |S| + r$, we obtain $q \geq \frac{rt}{d-t} + r = \frac{rd}{d-t}$. Thus, $q \geq \lceil \frac{rd}{d-t} \rceil$.

We claim that there are at least $\lceil \frac{rd}{d-t} \rceil$ indices i such that $t_i < d$. Otherwise, there are at most $\lceil \frac{rd}{d-t} \rceil - 1$ values of i such that $t_i < d$, which means that there are at least $q - \left(\lceil \frac{rd}{d-t} \rceil - 1 \right)$ values of i such that $t_i \geq d$. Because G is t-edge-connected, we have $t_i \geq t$ for each $1 \leq i \leq q$. These facts imply

$$\begin{aligned} d|S| &\geq \sum_{i=1}^{q} t_i \geq d\left[q - \left(\left\lceil \frac{rd}{d-t} \right\rceil - 1 \right) \right] + t\left(\left\lceil \frac{rd}{d-t} \right\rceil - 1 \right) \\ &= dq - (d-t)\left(\left\lceil \frac{rd}{d-t} \right\rceil - 1 \right) \\ &> dq - (d-t)\frac{rd}{d-t} = d(q-r) \\ &\geq d|S|, \end{aligned}$$

which is a contradiction. Here we used the inequality $x > \lceil x \rceil - 1$ for any real number x.

Let $p = \lceil \frac{rd}{d-t} \rceil$. Without loss of generality, assume that $t_i < d$ for $1 \leq i \leq p$. By Theorem 2 in [15] (see also Lemma 3.1.14), we obtain $\lambda_1(\mathcal{O}_i) \geq \rho(d)$ for $1 \leq i \leq p$. This fact and Interlacing Theorem 3.1.6 imply

$$\lambda_p(G) \geq \lambda_p(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_p) \geq \min_{1 \leq i \leq p} \lambda_1(\mathcal{O}_i) \geq \rho(d),$$

which finishes the proof. □

We are now ready to present the proof of Theorem 3.1.1.

Proof. For $x \in \{t, t+1\}$, we have $p = \lceil \frac{rd}{d-x} \rceil$ if and only if $r = \lfloor \frac{(d-x)p}{d} \rfloor$. This and the previous lemma imply the desired result. □

Corollary 3.1.8 (Cioabă, Gregory and Haemers [15]). *Let p be an integer with $p \geq 3$. If G is a connected d-regular graph of order n such that $\lambda_p(G) < \rho(d)$, then $\alpha'(G) > \frac{n-p+1}{2}$.*

Proof. Take $t = 1$ in Theorem 3.1.1. □

Corollary 3.1.9. *Let d and t be two integers with $d \geq 3$ and $t \leq d-2$. If G is a t-edge-connected d-regular graph of order n such that*

$$\rho(d) > \begin{cases} \lambda_{\lceil \frac{2d}{d-t} \rceil} & \text{if } d \equiv t \mod (2) \\ \lambda_{\lceil \frac{2d}{d-(t+1)} \rceil} & \text{if } d \equiv t+1 \mod (2) \end{cases}$$

then $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.

Proof. Take $r = 2$ in Lemma 3.1.7. □

When $t \in \{d-2, d-3\}$, the previous result states that a t-edge-connected d-regular graph with $\lambda_d(G) < \rho(d)$ must have $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$. In particular, because every connected 4-regular graph is 2-edge-connected, we deduce that if the fourth largest eigenvalue of a 4-regular graph is less than $\rho(4)$, then the matching number of the graph is $\lfloor \frac{n}{2} \rfloor$. The result of Cioabă, Gregory, and Haemers from [15] states for all d that $\lambda_3(G) < \rho(d)$ implies $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$; our result improves this when $d = 4$. In Figure 3.1, we present an example of a 2-edge-connected 4-regular graph H_1 having $\lambda_4(H_1) < \rho(4) = 1 + \sqrt{7} = \lambda_3(H_1)$; our result applies here, but the earlier result does not.

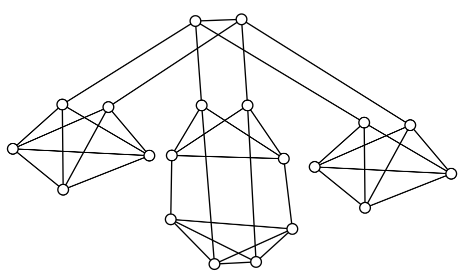

Figure 3.1: 2-edge-connected 4-regular graph with $\lambda_4 = 2 < \rho(4) = 1 + \sqrt{7} = \lambda_3$

In Figure 3.2, we present a 2-edge-connected 5-regular graph satisfying $\lambda_5(H_2) = 2 < \rho(5) = 1 + \sqrt{13} = \lambda_3(H_2)$. The existence of the perfect matching in these graphs will follow by taking edge-connectivity into account and using Corollary 3.1.9, but it cannot be deduced using the results from [15]. Similar examples can be constructed for larger values of d.

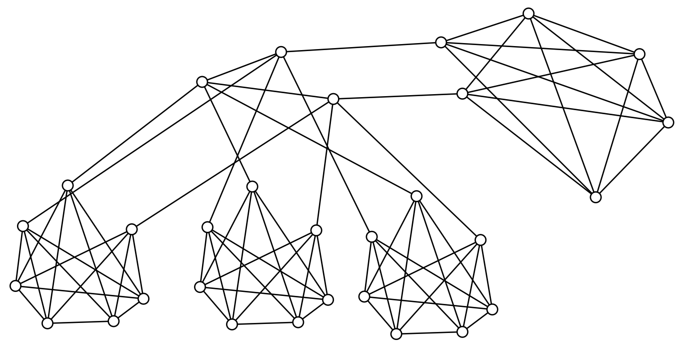

Figure 3.2: 2-edge-connected 5-regular graph with $\lambda_5 = 2 < \rho(5) = 1 + \sqrt{13} = \lambda_3$

Corollary 3.1.10. *If G is a d-regular $(d-2)$-edge-connected graph such that $\lambda_p(G) < \rho(d)$, then*

$$\alpha'(G) > \frac{n + \lfloor -\frac{2p}{d} \rfloor}{2}.$$

Proof. Take $t = d - 2$ in Theorem 3.1.1. □

We will show for each d that this result is best possible for infinitely many values of p by presenting examples of d-regular $(d-2)$-edge-connected graphs H with $\lambda_p(H) = \rho(d)$ and $\alpha'(G) = \frac{n+\lfloor -\frac{2p}{d} \rfloor}{2}$.

We prove Theorem 3.1.2, which relates the number of balloons of a regular graph to its eigenvalues.

When $d \geq 1$ is an odd integer, let B_d denote the unique graph on $d+2$ vertices having one vertex of degree $d-1$ and $d+1$ vertices of degree d. Equivalently, B_d is the complement of the disconnected graph on $d+2$ vertices with $\frac{d-1}{2}$ components equal to K_2 and one component equal to P_3, the path on three vertices.

Lemma 3.1.11. *If $\theta(d)$ denotes the largest solution of the equation*

$$x^3 - (d-1)x^2 - 2dx + d - 1 = 0,$$

then the spectral radius of B_d is $\theta(d)$.

Proof. Partition the vertex set of B_d in three parts: the vertex of degree $d-1$, its neighbours and the remaining two vertices. This partition is equitable, and its quotient matrix is

$$\begin{bmatrix} 0 & d-1 & 0 \\ 1 & d-3 & 2 \\ 0 & d-1 & 1 \end{bmatrix}.$$

The characteristic polynomial of this quotient matrix equals $P(x) = x^3 - (d-1)x^2 - 2dx + d - 1$. Because the partition is equitable, we conclude that the spectral radius of B_d equals the largest root of this polynomial (see [27]). This finishes the proof. □

A result from [14] and a simple manipulation yield the following bounds

$$d - \frac{1}{d+2} + \frac{1}{(d+2)^2} < \theta(d) < d - \frac{1}{d+2} + \frac{1}{d(d+2)}. \tag{3.5}$$

Lemma 3.1.12. *If H is a connected graph of order m with $m-1$ vertices of odd degree d and one vertex of degree $d-1$, then $\lambda_1(H) \geq \theta(d)$ with equality if and only if $H = B_d$.*

Proof. If $m = d+2$, then $H = B_d$. We will show that $\lambda_1(H) > \theta(d)$ for any connected graph H if $m > d+2$. Because the sum of the degrees of H is $dm-1$ and d is odd, we conclude that m is odd.

The average degree of H is $\frac{(m-1)d+1(d-1)}{m} = d - \frac{1}{m}$. If $m \geq d+4$, then (3.5) implies that

$$\lambda_1(H) > d - \frac{1}{m} \geq d - \frac{1}{d+4} > d - \frac{d-1}{d^2+2d} > \theta(d)$$

for $d \geq 5$. Thus, the lemma is proved for $d \geq 5$ and $m \geq d+4$. If $d \geq 5$ and $m < d+4$, then $m = d+2$ because n is odd. This finishes the case $d \geq 5$.

When $d = 3$, we have $\lambda_1(B_3) = 2.855... < 2.86$. If $m \geq 9$, then $\lambda_1(H) > 3 - \frac{1}{9} > 2.89 > \lambda_1(B_3)$. Because m is odd, the only remaining case is $m = 7$. In this case, each graph H has one vertex u of degree 2, and six vertices of degree 3. We have two cases depending on whether the neighbours of u are adjacent.

If the neighbours of u are adjacent, then H is the graph pictured in Figure 3.3 and its spectral radius is greater than 2.91, which exceeds $\lambda_1(B_3)$.

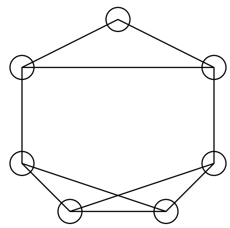

Figure 3.3: The neighbours of the vertex of degree 2 are adjacent

If the neighbours of u are not adjacent, then H is one of the three graphs pictured in Figure 3.4. The spectral radii of these graphs are greater than 2.9, which exceeds $\lambda_1(B_3)$.

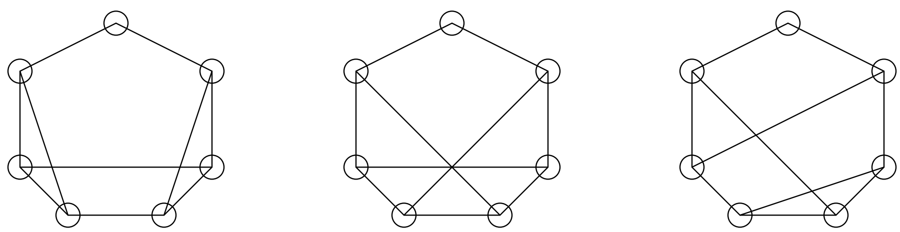

Figure 3.4: The neighbours of the vertex of degree 2 are not adjacent

$\square$

Note that every balloon of a d-regular graph satisfies the conditions of the previous lemma.

Proof of Theorem 3.1.2. We prove the contrapositive: For $k \geq 3$ if G contains k balloons, then $\lambda_k(G) \geq \theta(d)$.

Let $H_1, \ldots, H_k$ be k balloons of G. These balloons are pairwise disjoint, and $H_1 \cup \cdots \cup H_k$ is an induced subgraph of G. Note that $H_1 \cup H_2$ might not be an induced subgraph of G when G contains exactly two balloons. The previous lemma and Interlacing Theorem 3.1.6 imply that

$$\lambda_k(G) \geq \lambda_k(H_1 \cup \cdots \cup H_k) \geq \min_{1 \leq i \leq k} (\lambda_1(H_i)) \geq \theta(d).$$

$\square$

Fixing d, for some values of k, we construct examples of d-regular graphs H containing exactly k balloons and having $\lambda_k(H) = \theta(d)$. The vertex of B_d having degree $d-1$ is called the neck of B_d.

Let $m \geq d-1$ be an even integer and let $k = 2m$. Consider a connected $(d-2)$-regular graph X on m vertices. For each vertex x of X, create two new vertices x_1 and x_2 adjacent to x. Identify each of x_1 and x_2 with the neck of a copy of B_d. The resulting graph G_1 is connected, d-regular, has $m + 2m(d+2)$ vertices, and contains k balloons.

Removing the m vertices of X yields a disconnected graph with $2m$ components, each isomorphic to B_d. Interlacing Theorem 3.1.6 implies

$$\lambda_{k-m}(G_1) \geq \lambda_{k-m}(2mB_d) = \theta(d) \geq \lambda_k(G_1) \geq \lambda_k(2mB_d) = \theta(d),$$

which means $\lambda_k(G_1) = \theta(d)$.

For $d = 3$, we can construct a 3-regular graph G_k with $\lambda_k(H) = \theta(3)$ and $b(H) = k$ for any $k \geq 3$. Consider a tree of order n whose degrees are 1 or 3. If k is the number of leaves, then $2(n-1) = k + 3(n-k)$; thus $n = 2k-2$. Identify each leaf with the neck of a copy of B_3. If G_k is the resulting graph, then G_k is 3-regular and $b(H) = k$. Remove the vertices adjacent to a neck of a balloon. There are $t \leq n-k = k-2$ such vertices. By Interlacing Theorem 3.1.6, we obtain

$$\theta(3) = \lambda_2(kB_3) \geq \lambda_{k-t}(kB_3) \geq \lambda_k(G_k) \geq \lambda_k(kB_3) = \theta(3),$$

and thus $\lambda_k(G_k) = \theta(3)$.

Each of these examples can be transformed into an infinite family of examples by replacing one balloon B_d or B_3 by a larger balloon.

In this section, we give a proof of Theorem 3.1.4. We use the following notation. Let C_m denote the cycle on m vertices. If m is even, we denote by M_m the 1-regular graph on m vertices, i.e. a perfect matching on m vertices. Also, $\overline{G}$ will denote the complement of a graph G. If H_1 and H_2 are vertex disjoint graphs, their join $H_1 \vee H_2$ is the graph obtained from H_1 and H_2 by joining each vertex of H_1 with each vertex of H_2.

Let d and t be two integers of the same parity with $d > t \geq 1$. Define the graph $B_{d,t}$ as follows:

$$B_{d,t} = \begin{cases} K_{d+1-t} \vee \overline{M_t} & \text{if } d \text{ and } t \text{ are even} \\ \overline{M_{d+2-t}} \vee \overline{C_t} & \text{if } d \text{ and } t \text{ are odd} \end{cases} \tag{3.6}$$

Lemma 3.1.13. *The spectral radius of $B_{d,t}$ equals*

$$\lambda_1(B_{d,t}) = \begin{cases} \frac{d-2+\sqrt{(d+2)^2-4t}}{2} < d - \frac{t}{d+2} & \text{if } d \text{ and } t \text{ are even} \\ \frac{d-3+\sqrt{(d+3)^2-4t}}{2} < d - \frac{t}{d+3} & \text{if } d \text{ and } t \text{ are odd} \end{cases} \tag{3.7}$$

Proof. We prove this lemma only in the case when d and t are even. The proof of the other case is similar and is omitted.

Let V_1 and V_2 be the subsets of vertices of degree d and degree $d-1$ in $B_{d,t}$, respectively. From the definition of $B_{d,t}$, it is easy to see that $|V_1| = d+1-t$ and $|V_2| = t$. The partition $V(B_{d,t}) = V_1 \cup V_2$ is equitable, and its quotient matrix is

$$\begin{bmatrix} d-t & t \\ d-t+1 & t-2 \end{bmatrix}.$$

The characteristic polynomial of the quotient matrix is $x^2-(d-2)x-2d+t$. Because the partition is equitable, we deduce that the spectral radius of $B_{d,t}$ equals the largest root of this polynomial, which is $\frac{d-2+\sqrt{(d+2)^2-4t}}{2}$. The inequality $\frac{d-2+\sqrt{(d+2)^2-4t}}{2} < d - \frac{t}{d+2}$ follows easily using the inequality $\sqrt{x^2+a} < x + \frac{a}{2x}$ for $x > 0$. This finishes the proof. $\square$

Note that $\lambda_1(B_{d,d-2})$ equals $\rho(d)$ from Theorem 3.1.1. The following result extends Theorem 2 from [15].

Lemma 3.1.14. *Let d and t be two integers of the same parity with $d > t \geq 1$. If H is a graph of order m such that $\Delta(H) = d$ and $2e(H) = dm - t$, then $\lambda_1(H) \geq \lambda_1(B_{d,t})$. Equality occurs if $H = B_{d,t}$ when d and t are even and if $H = \overline{M_{d+2-t}} \vee \overline{C_{t_1} \cup \cdots \cup C_{t_l}}$ where $t_1 + \cdots + t_l = t$, when d and t are odd.*

Proof. The average degree of H is $\frac{dm-t}{m} = d - \frac{t}{m}$. This implies $\lambda_1(H) \geq d - \frac{t}{m}$.

Let $V(H) = V_1 \cup V_2$ be a partition of the vertex set of H. Let $n_i = |V_i|$, and denote by e_i the number of edges of the subgraph induced by V_i for $1 \leq i \leq 2$. Let $e_{12} = e(V_1, V_2)$. The quotient matrix of this partition is

$$\tilde{A} = \begin{bmatrix} \frac{2e_1}{n_1} & \frac{e_{12}}{n_1} \\ \frac{e_{12}}{n_2} & \frac{2e_2}{n_2} \end{bmatrix}. \tag{3.8}$$

Eigenvalue interlacing (see [10, 27] or the last part of Section 1) implies

$$\lambda_1(H) \geq \lambda_1(\tilde{A}) = \frac{e_1}{n_1} + \frac{e_2}{n_2} + \sqrt{\left(\frac{e_1}{n_1} - \frac{e_2}{n_2}\right)^2 + \frac{e_{12}^2}{n_1 n_2}}, \tag{3.9}$$

with equality if the partition $V(H) = V_1 \cup V_2$ is equitable.

If $d > t \geq 1$ are both even and $m \geq d+2$, then by Lemma 3.1.13, we obtain $\lambda_1(H) \geq d - \frac{t}{m} \geq d - \frac{t}{d+2} > \lambda_1(B_{d,t})$. Thus, the only remaining case is $m = d+1$.

If $m = d+1$, then H is a graph obtained by deleting $\frac{t}{2}$ distinct edges from K_{d+1}. The graph H has at most t vertices of degree less than $d-1$, and thus it contains at least $d+1-t$ vertices of degree d. Let V_1 be a set of $d+1-t$ vertices of degree d, and let $V_2 = V(H) \setminus V_1$. Using (3.9), we obtain $\lambda_1(H) \geq \lambda_1(B_{d,t})$, with equality if $H = B_{d,t}$.

If $d > t \geq 1$ are both odd and $m \geq d+3$, then by Lemma 3.1.13, we obtain $\lambda_1(H) \geq d - \frac{t}{m} \geq d - \frac{t}{d+3} > \lambda_1(B_{d,t})$.

If $m = d+2$, then H is a graph of maximum degree d obtained by deleting $\frac{d+2+t}{2}$ distinct edges from K_{d+2}. The graph H has at most t vertices of degree $d-1$, and thus it contains at least $d+2-t$ vertices of degree d. Let V_1 be a set of $d+2-t$ vertices of degree d and let $V_2 = V(H) \setminus V_1$. Let $2r$ denote the maximum number of vertices of V_1 which induce a 1-regular graph in $\overline{H}$.

Case 1. $r = (d+2-t)/2$.

In this case, $e_1 = \frac{(d+2-t)d}{2}$, $e_{12} = (d+2-t)t$ and $e_2 = \frac{t(t-3)}{2}$. We use inequality (3.9) to obtain $\lambda_1(H) \geq \lambda_1(B_{d,t})$. Equality occurs if the partition $V(H) = V_1 \cup V_2$ is equitable. This happens when $H = \overline{M_{d+2-t}} \vee \overline{C_{t_1} \cup \cdots \cup C_{t_l}}$ where $t_1 + \cdots + t_l = t$.

Case 2. $r = 0$

In this case, $e_1 = \binom{d+2-t}{2}$, and each vertex of V_1 is adjacent to all but one vertex of V_2. Thus, $e_{12} = n_1(n_2 - 1) = (d + 2 - t)(t - 1)$. This and $2e = d(d + 2) - t$ imply that $e_2 = e - e_1 - e_{12} = \frac{d(d+2)-t}{2} - \frac{(d+2-t)(d+1-t)}{2} - (d + 2 - t)(t - 1) = \frac{d+t^2-4t+2}{2}$.

In this case, the characteristic polynomial of $\tilde{A}$ is the following

$$P_{\tilde{A}}(x) = x^2 - \frac{dt + d + 2 - 3t}{t}x + \frac{d^2 + 2d + t^2 - 3dt - t}{t}.$$

We obtain

$$P_{\tilde{A}}(x) = (x^2 + (3 - d)x + t - 3d) + \frac{d(d + 2) - t - (d + 2)x}{t}.$$

Thus,

$$\begin{aligned}
P_{\tilde{A}}(\lambda_1(B_{d,t})) &= \frac{d+2}{t}\left(d - \frac{t}{d+2} - \lambda_1(B_{d,t})\right) \\
&= \frac{d+2}{t}\left(d - \frac{t}{d+2} - \frac{d - 3 + \sqrt{(d+3)^2 - 4t}}{2}\right) \\
&< 0,
\end{aligned}$$

where the last inequality follows by straightforward calculations as $t < d+2$. We conclude that the largest root of $\tilde{A}$ is larger than $\lambda_1(B_{d,t})$. By eigenvalue interlacing, this means $\lambda_1(H) > \lambda_1(B_{d,t})$.

Case 3. $1 \le r \le (d - t)/2$

Consider the partition of $V(H)$ into three parts: the $2r$ vertices inducing a $\overline{M_{2r}}$ in H, the other $d + 2 - t - 2r$ vertices of degree d in V_1, and the remaining t vertices. The quotient matrix of this partition is

$$A_3 = \begin{bmatrix} 2r - 2 & d + 2 - t - 2r & t \\ 2r & d + 1 - t - 2r & t - 1 \\ 2r & \frac{(d+2-t-2r)(t-1)}{t} & t - 1 - \frac{2r+3t-d-2}{t} \end{bmatrix}$$

Dividing the characteristic polynomial $P_{A_3}(x)$ of A_3 by $x^2 + (3 - d)x + t - 3d$, we obtain

$$P_{A_3}(x) = (x^2 + (3 - d)x + t - 3d)\left(x + \frac{2t + 2r - d - 2}{t}\right) + \frac{(d + 2 - t - 2r)(x - d)}{t} \tag{3.10}$$

Plugging in $x = \lambda_1(B_{d,t})$, we get $P_{A_3}(\lambda_1(B_{d,t})) = \frac{(d+2-t-2r)(\lambda_1(B_{d,t})-d)}{t}$. This expression is negative because $r \leq \frac{d-t}{2}$ and $\lambda_1(B_{d,t}) < d$. Thus, the largest root of $P_{A_3}(x)$ is greater than $\lambda_1(B_{d,t})$. Eigenvalue interlacing (see [10, 27]) implies $\lambda_1(H) > \lambda_1(B_{d,t})$ and finishes the proof. □

We can now prove Theorem 3.1.4.

Proof of Theorem 3.1.4. We can assume that G has edge-connectivity t. If $b_t(G) \geq p$, there exist at least p disjoint connected subgraphs $X_1, \ldots, X_p$ of G that satisfy the conditions of Lemma 3.1.14. Because $p \geq 3$, $X_1 \cup \cdots \cup X_p$ is an induced subgraph of G. Lemma 3.1.14 and Interlacing Theorem 3.1.6 imply

$$\lambda_p(G) \geq \lambda_p(X_1 \cup \cdots \cup X_p) \geq \min_{1 \leq i \leq p} \lambda_1(X_i) \geq \lambda_1(B_{d,t}).$$

□

We present now a construction for fixed d showing that Theorem 3.1.4 is best possible for infinitely many values of p.

Let t be an integer of the same parity as d with $d > t \geq 1$. Also, let $p \geq 3$ be a positive integer such that $\frac{pt}{d}$ is also an integer. Let Y be a t-edge-connected bipartite graph with colour classes P and Q such that $|P| = p$ and $|Q| = \frac{pt}{d} < k$. Assume also that each vertex in the colour class P has degree t and each vertex in Q has degree d. See [50] for a proof of the existence of such graphs.

For each vertex $x \in P$, consider its neighbours $x_1, \ldots, x_t \in Q$. Remove x and add t new vertices $y_1, \ldots, y_t$ such that y_i is adjacent to x_i for each $1 \leq i \leq t$. Identify $y_1, \ldots, y_t$ with the t vertices of degree $d-1$ from a copy of $B_{d,t}$. The resulting graph H is d-regular, t-edge-connected (see [50] for a short proof of this fact) and has $q + p(d - 2 + \epsilon)$ vertices, where $\epsilon = 3$ if d is even and $\epsilon = 4$ if d is odd.

Removing Q from H creates a disconnected graph $H - Q$ having p components $B_{d,t}$. Interlacing Theorem 3.1.6 implies

$$\lambda_1(B_{d,t}) = \lambda_{p-\frac{pt}{d}}(H - Q) \geq \lambda_p(H) \geq \lambda_p(H - Q) = \lambda_1(B_{d,t}). \tag{3.11}$$

Thus, $\lambda_p(H) = \lambda_1(B_{d,t})$ and also, $b_t(H) = p$.

In this section, we show that Theorem 3.1.1 is best possible when $t = d-2$ by presenting examples of d-regular graphs H of edge-connectivity $d-2$ with $\lambda_p(H) = \rho(d)$ and $\alpha'(H) = \frac{n+\lfloor -\frac{2p}{d} \rfloor}{2}$ for infinitely many values of p.

Let d and s be two integers with $d \geq 3$ and $s \geq 1$. Let $p = (d-2)s$ and $q = ds$. Our construction consists of the graphs H presented at the end of the previous section in the special case $t = d-2$.

Removing the vertices of Q creates p disjoint copies $B_{d,d-2}$. Because $B_{d,d-2}$ has an odd number of vertices, it follows that $o(H-Q) = p$. Using (3.3), the matching number of H will be at least $\frac{n-p+q}{2} = \frac{n-ds+(d-2)s}{2} = \frac{n-2s}{2}$. It is actually easy to see that $\alpha'(H) = \frac{n-2s}{2} = \frac{n+\lfloor -\frac{2p}{d} \rfloor}{2}$. By Interlacing Theorem 3.1.6, we obtain $\lambda_p(H) = \lambda_1(B_{d,d-2}) = \rho(d)$ as claimed.

In Figure 3.5, we illustrate this construction when $d = 4$ and $Y = K_{4,2}$.

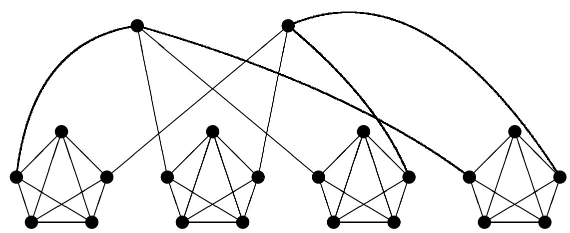

Figure 3.5: 4-regular graph with edge-connectivity 2 and $\lambda_4 = \rho(4)$

Plesnik [54] (see also Exercise 30, Section 7 in Lovász [39]) proved that a graph obtained by removing $d-1$ edges from a d-regular $(d-1)$-edge-connected contains a perfect matching. This implies that a d-regular $(d-1)$-edge-connected graph is matching covered, i.e. each edge is contained in a perfect matching. It would be interesting to determine sharp relations between this property and the eigenvalues of a graph.

3.2 Eigenvalues and Matching

A lot of research in graph theory over the last 40 years was stimulated by a classical result of Fiedler [25], stating that $\kappa(G) \geq \mu_2(G)$ for a non-complete graph G, where $\kappa(G)$ is the connectivity of G and $\mu_2(G)$ is the second smallest eigenvalue of the Laplacian matrix.

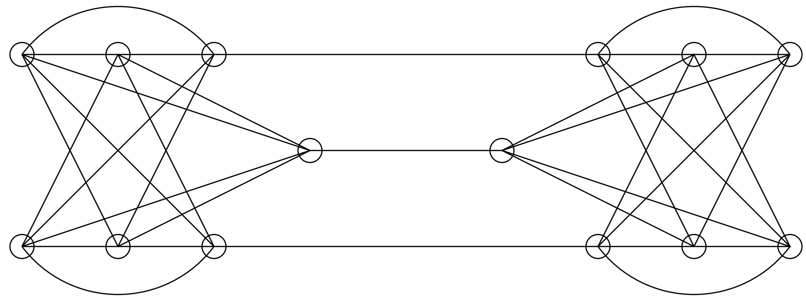

Figure 3.6: $Y_{5,3}$ is a 5-regular graph with $\lambda_2(Y_{5,3}) = \rho(5,3) = \frac{1+\sqrt{57}}{2}$

Let G be a simple graph with n vertices, and let A be the adjacency matrix of G. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A.

In this section, we study the problem of finding the weakest hypothesis on the second largest eigenvalue λ_2 for a d-regular graph G to guarantee that G is l-edge-connected.

In 2010, Cioaba [12] proved that if $\lambda_2 < d - \frac{2(l-1)}{d+1}$, then G is l-edge-connected. However, this result is not sharp; he proved stronger results when l is equal to 2 or 3.

Theorem 3.2.1. (Cioaba [12]) *Let d be an odd integer at least 3 and let $\pi(d)$ be the largest root of $x^3 - (d-3)x^2 - (3d-2)x - 2 = 0$. If G is a d-regular graph such that $\lambda_2 < \pi(d)$, then $\kappa'(G) \geq 2$.*

Theorem 3.2.2. (Cioaba [12]) *If G is d-regular graph such that $\lambda_2(G) < \frac{d-3+\sqrt{(d+3)^2-16}}{2}$, then $\kappa'(G) \geq 3$.*

Interestingly, the sharpness examples are derived by combining two graphs from Chapter 2. First, we study the sharpest examples.

Before studying the examples, we recall basic definitions.

A *disconnecting set* in a multigraph G is a set $F \subseteq E(G)$ such that $G - F$ is disconnected. A multigraph is *k-edge-connected* if every disconnecting set has at least k edges. The *edge-connectivity* $\kappa'(G)$ is $\max\{k : G \text{ is } \text{k-edge-connected}\}$ For vertex sets S and T, we write $[S,T]$ for the set of edges from S to T. An *edge cut* is a set of the form $[S,\overline{S}]$, where $\emptyset \neq S \subset V(G)$

Now, we construct the sharpenss examples. First, for even positive integer t, let M_t be the set of $\frac{t}{2}$ disjoint edges. Note that there are t vertices in M_t. For odd positive integer d and l with $l \leq d-2$, we define the graph $H_{d,l} = \overline{M_{d+2-l}} \vee \overline{C_l}$. Note that $H_{d,l}$ has exactly $d+2-l$ vertices with degree d and exactly l vertices with degree $d-1$. We construct $Y_{d,l}$ by taking two disjoint copies of $H_{d,l}$ and adding two disjoint edges between the vertices of degree $d-1$ in different copies of $H_{d,l}$. Figure 3.6 descirbes $Y_{5,3}$.

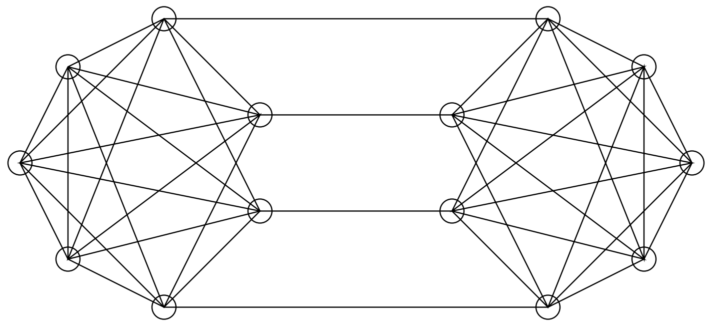

Figure 3.7: $Y_{6,4}$ is a 6-regular graph with $\lambda_2(Y_{6,4}) = \rho(6,4) = \frac{3+\sqrt{59}}{2}$

Similarly, for even positive integers d and l with $l \le d-2$, we can define $Y_{d,l}$. Consider the graph $H_{d,l} = K_{d+1-t} \vee \overline{M_l}$. Note that for even positive integer d and l, $H_{d,l}$ has exactly $d+1-l$ vertices with degree d and exactly l vertices with degree $d-1$. We construct $Y_{d,l}$ by taking two disjoint copies of $H_{d,l}$ and adding two disjoint edges between the vertices of degree $d-1$ in different copies of $H_{d,l}$. Figure 3.7 descirbes $Y_{6,4}$.

Lemma 3.2.3. *The graph $H_{d,l}$ is a connected d-regular with $\kappa'(H_{d,l}) = l$.*

Proof. By construction, $H_{d,l}$ is d-regular, connected, and $\kappa'(H_{d,l}) \le l$. To show that $\kappa'(H_{d,l}) = l$, consider $F \subseteq E(H_{d,l})$ with size $|F| < l$. Note that $l < d-1$ since $l \le d-2$. Since $H_{d,l}$ is $(d-1)$-edge-connected by Lemma 2.2.12 and $l < d-1$, we can have a path from any vertex in a copy of $H_{d,l}$ to any vertex in the copy. Since $|F| < l$, there is a way from any vertex in one copy of $H_{d,l}$ to any vertex in another copy of $H_{d,l}$. Thus, $Y_{d,l}$ is l-edge-connected, which implies that $\kappa(Y_{d,l}) = l$. $\square$

Now, we can determine the second largest eigenvalue of $Y_{d,l}$.

First, for odd positive integer d, consider an equitable partition into four parts of sizes $d+2-l$, l, l, $d+2-l$ with the following quotient matrix :

$$\tilde{Y_{d,l}} = \left\{ \begin{pmatrix} d-l & l & 0 & 0 \\ d+2-l & l-3 & 1 & 0 \\ 0 & 1 & l-3 & d+2-l \\ 0 & 0 & l & d-l \end{pmatrix} \right. \tag{3.12}$$

The characteristic polynomial of the matrix (3.12) is equal to

$$P_{\tilde{Y_{d,l}}} = (x-d)(x+2)(x^2-(d-4)x+2l-4d).$$

Thus, the eigenvalues of $\tilde{Y_{d,l}}$ are d, -2 and $\frac{d-4\pm\sqrt{(d+4)^2-8l}}{2}$. To simplify our notation, let $\rho(d,l) = \frac{d-4+\sqrt{(d+4)^2-8l}}{2}$. Note that $\rho(d,l)$ is the largest root of $T(x) = (x^2-(d-4)x+2l-4d)$.

Similarly, for even positive integer d, consider an equitable partition into four parts of sizes $d+1-l$, l, l, $d+1-l$ with the following quotient matrix :

$$\tilde{Y_{d,l}} = \left\{ \begin{pmatrix} d-l & l & 0 & 0 \\ d+1-l & l-2 & 1 & 0 \\ 0 & 1 & l-2 & d+1-l \\ 0 & 0 & t & d-l \end{pmatrix} \right. \tag{3.13}$$

The characteristic polynomial of the matrix (3.13) is equal to

$$P_{\tilde{Y_{d,l}}} = (x-d)(x+1)(x^2-(d-3)x+2l-3d).$$

Thus, the eigenvalues of $\tilde{Y_{d,l}}$ are d, -1 and $\frac{d-3\pm\sqrt{(d+3)^2-8l}}{2}$. For even positive integer d, let $\rho(d,l) = \frac{d-3+\sqrt{(d+3)^2-8l}}{2}$. Note that $\rho(d,l)$ is the largest root of $T(x) = (x^2-(d-3)x+2l-3d)$.

Lemma 3.2.4. *The second largest eigenvalue of $Y_{d,l}$ is equal to $\rho(d,l)$.*

Proof. First, we consider when d is odd. Since the previous partition of $V(Y_{d,l}$ into the four parts are equitable, we have that the four eigenvalues of $\tilde{Y_{d,l}}$ are also eigenvalues of $Y_{d,l}$. Clearly, the second largest eigenvalue of $\tilde{Y_{d,l}}$ is $\rho(d,l)$.

Let $W \subset \mathbb{R}^{2d+4}$ be the subspace of vertors which are constant on each part of the four equitable partition. The lifted eigenvectors corresponding to the four roots of $P_{\tilde{Y_{d,l}}}$ form a basis for W. The remaining eigenvectors in a basis of eigenvectors for $Y_{d,l}$ can be chosen to be perpendicular to the vectors in W. Thus, they may be chosen to be perpendicular to the characteristic vectors of the parts in the four-part equitable partition since these characteristic vectors form a basis for W. This implies that these eigenvectors will correspond to the non-trivial eigenvalues of the grahp obtained as a disjoint union of $2\overline{M_{d+2-l}}$ and $2\overline{C_l}$. Thus, we determine $Y_{d,l} = \rho(d,l)$.

Similarly, we can determine $\tilde{Y_{d,l}} = Y_{d,l} = \rho(d,l)$ when d is even. $\square$

Conjecture 3.2.5. *If G is a d-regular graph such that for $l \geq 2$,*

$$\lambda_2(G) < \rho(d,l),$$

then $\kappa'(G) \geq l+1$.

Notice that

$$\frac{d-4+\sqrt{(d+4)^2-8(l+1)}}{2} < \frac{d-3+\sqrt{(d+3)^2-8l}}{2} < \frac{d-4+\sqrt{(d+4)^2-8(l-1)}}{2} \tag{3.14}$$

Proposition 3.2.6. *Assume that G is a d-regular graph with $\kappa'(G) \leq l$ for $l \geq 2$. If there exists a subset $S \subseteq V(G)$ with $|[S,\overline{S}]| = \kappa'(G)$ and both $|S|$ and $|\overline{S}|$ are at least $d+4$ when d is odd and at least $d+3$ when d is even, respectively, then*

$$\lambda_2(G) \geq \rho(d,l).$$

Proof. Let $\kappa'(G) = t$. By hypothesis, there exsit vertex subsets V_1 and V_2 in $V(G)$ such that $|[V_1,V_2]| = t$ and $V_1 \cup V_2 = V(G)$. Let $G[V_1] = G_1$ and $G[V_2] = G_2$. Suppose that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$. We may assume that $n_1 \leq n_2$. If d is odd, then it follows that $n_1 \geq d+2$ and n_i is odd for each $i = 1,2$.

Consider the partition of $V(G)$ into V_1 and V_2. The quotient matrix of the partition is

$$\begin{pmatrix} d-\frac{t}{n_1} & \frac{t}{n_1} \\ \frac{t}{n_2} & d-\frac{t}{n_2} \end{pmatrix}$$

and its eigenvalues are d and $d-\frac{t}{n_1}-\frac{t}{n_2}$. Eigenvalue interlacing, $n_2 \geq n_1 \geq d+4$, and $l \geq t$ imply that $\lambda_2(G) \geq d-\frac{t}{n_1}-\frac{t}{n_2} \geq d-\frac{2t}{d+4} > \frac{2l}{d+4} > \rho(d,l)$.

Similarly, we can prove for even positive integer d and in this case, it is ture for $n_1 \geq d+3$. $\square$

This gives a partial positive answer to the Conjecture 3.2.6.

If we can prove the remaining cases $n_1 = d+2$ for odd d and $n_1 \leq d+2$ for even d, then the conjecture is true, but the cases give a lot of analytic cases. If l is small, then we might handle the cases, but otherwise, we need to develop new techniques.

Chapter 4

Extremal Problems for Regular Graphs

In this chapter, we deal with three extremal problems in regular graphs.

In the first section, we study the Chinese Postman Problem in a connected $(2r+1)$-regular graph with n vertices, and the problem is equivalent to finding a smallest spanning subgraph in which all vertices have odd degree. We establish an upper bound for the solution in cubic graphs and characterize when equality holds.

In the second section, we prove that if G is a connected 4-regular graph with n vertices, then the vertices of G can be covered using at most $\lceil \frac{n}{7} \rceil$ disjoint paths. In addition, we propose several open questions.

In the last section, we study relationships between average connectivity and other graph parameters. We prove an upper bound on average connectivity in terms of matching number. We also establish a lower bound on the the average edge-connectivity of a connected cubic graph with n vertices, and we characterize when equality holds in this bound.

4.1 Chinese Postman Problem

The Chinese Postman Problem in a graph is the problem of finding a shortest closed walk traversing all the edges. In a $(2r+1)$-regular graph, the problem is equivalent to finding a smallest spanning subgraph in which all vertices have odd degree. In this section, we establish an upper bound for the solution in cubic graphs and characterize when equality holds.

The Chinese Postman Problem was introduced in the early 1960s by the Chinese mathematician Guan Meigu. Roughly speaking, a postman wishes to travel along every road in a city in order to deliver letters, with the least possible total distance. More precisely, a *postman tour* in a connected graph G is a closed walk containing all the edges of G. The problem is to find a shortest postman tour in G. An optimal postman tour in a connected graph G is a shortest closed walk traversing all edges in G. Since all edges of G must be included, we are interested only in the additional length needed. Let $p(G) = l - |E(G)|$, where l is the minimum length of a postman tour; we call $p(G)$ the parity number of G.

Since a postman tour is an Eulerian supergraph obtained by repeating some edges, $p(G)$ equals the minimum number of edges in a parity subgraph of G, where a parity subgraph is a spanning subgraph H of G such that $d_G(v) \equiv d_H(v) \pmod 2$ for every vertex v in G. In this section, we obtain the best upper bound on the parity number of a connected cubic graph with n vertices. We will show that a family that we introduced in Section 2 of Chapter 2 is the family of graphs having the largest parity numbers among cubic graphs, for appropriate congruence classes n.

In Section 2 of Chapter 2, we defined $\mathcal{F}_{n,r}$ to be the family of all connected $(2r+1)$-regular graphs with n vertices. We also defined a special graph B_r and a family H_r', which we now recall.

Example 4.1.1. Let B_r be the graph obtained from the complete graph K_{2r+3} by deleting a matching of size $r+1$ and deleting one more edge incident to the remaining vertex. This is the smallest graph in which one vertex has degree $2r$ and the others have degree $(2r+1)$. Thus B_r is the smallest possible balloon in a $(2r+1)$-regular graph. Note that deleting the vertex of degree $2r$ (the neck) from B_r leaves a subgraph having a perfect matching.

Let $\mathcal{T}_r'$ be the family of trees such that every non-leaf vertex has degree $2r+1$. Let $\mathcal{H}_r'$ be the family of $(2r+1)$-regular graphs obtained from trees in $\mathcal{T}_r'$ by identifying each leaf of such a tree with the neck in a copy of B_r. □

We proved that H_r' is the family of connected $(2r+1)$-regular graphs that have the most cut-edges among the graphs with a fixed number of vertices when the number of vertices is in an appropriate congruence class. For $r = 1$, we show that these graphs are also the graphs in $\mathcal{F}_{n,r}$ with largest parity number for given n. We conjecture that this also holds for larger r.

Conjecture 4.1.2. *If $G \in \mathcal{F}_{n,r}$, then* $p(G) \le \frac{(2r^2+3r-1)n-2(r+1)}{4r^2+4r-2} - 1$

The family H_r' shows that Conjecture 4.1.2 cannot be improved when n is in appropriate congruence classes.

In Section 2.1, we proved the following proposition about graphs in $\mathcal{H}'_r$.

Proposition 4.1.3. *Let $q_r = 2r^2 + 2r - 1$. For any n-vertex graph G in $\mathcal{H}'_r$,*

$$b(G) = \tfrac{(2r-1)n+2}{2q_r}, \quad \text{and} \qquad c(G) = \tfrac{r(n-2)-2}{q_r} - 1. \ ,$$

where $b(G)$ and $c(G)$ are the number of balloons in G and the number of cut-edges in G, respectively, and n satisfies appropriate congruence classes.

Lemma 4.1.4. *If G is regular of odd degree, then every cut-edge is in every parity subgraph.*

Proof. Let e be a cut-edge in G. By the Degree-Sum Formula, each component of $G - e$ has an odd number of vertices. Since a parity subgraph has odd degree at each vertex, the Degree-Sum Formula then implies that the parity subgraph must contain e. □

Since every edge of a tree is a cut-edge, we obtain the following corollary.

Corollary 4.1.5. *If G is a graph in $\mathcal{H}'_r$, and T is the tree obtained by shrinking each B_r in G to one vertex, then every parity subgraph of G contains T.*

Next, we determine the parity number of graphs in $\mathcal{H}'_r$.

Proposition 4.1.6. *If G is in $\mathcal{H}'_r$, then*

$$p(G) = \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1,$$

which reduces to $\frac{2n-5}{3}$ for cubic graphs.

Proof. Let T be the tree obtained by shrinking all the balloons in G. By Corollary 4.1.5, a parity subgraph must use all the edges in T. A parity subgraph of G must contain all cut-edges, which cover the necks of some balloons. A parity subgraph of G must add at least $r + 1$ more edges in each balloon (since B_r has $2r + 3$ vertices, and all have odd degree in G). Hence, $p(G) \geq c(G) + (r + 1)b(G) = \frac{r(n-2)-2}{q_r} - 1 + (r + 1)\frac{(2r-1)n+2}{2q_r} = \frac{(2r^2+3r-1)n-2(r+1)}{4r^2+4r-2} - 1$ by Proposition 4.1.3. By taking all edges of T plus a near perfect matching in each copy of B_r, equality is achieved. □

Definition 4.1.7. *An* r-graph[1] *is an r-regular multigraph G on an even number of vertices such that for every odd-sized subset X of $V(G)$, the number of edges with exactly one endpoint in X is at least r.*

Remark 4.1.8. *Note that if G is a 2-edge-connected cubic multigraph, then G is a 3-graph, since the Degree-Sum Formula forces $|[S,\overline{S}]| = 3$ for every odd-sized subset S of $V(G)$. More generally, if G is an $(r-1)$-edge-connected r-regular multigraph with even order, then G is an r-graph for the same reason. Also, every r-edge-colorable r-regular graph is an r-graph.*

We need a fundamental result about r-graphs due to Edmonds.

Theorem 4.1.9. (Edmonds [22]) *If G is an r-graph, then there is an integer p and a family $\mathcal{M}$ of perfect matchings such that each edge of G is contained in precisely p members of $\mathcal{M}$. (The members of $\mathcal{M}$ need not be distinct.)*

Lemma 4.1.10. *If G is a $2r$-edge-connected $(2r+1)$-regular multigraph, in which each edge e has weight $w(e)$, then there exists a perfect matching with weight at most $\frac{1}{2r+1}W$, where $W = \sum_{e\in E(G)} w(e)$. For cubic graphs, the bound reduces to $\frac{1}{3}W$.*

Proof. Let $\mathcal{M}$ be a family of perfect matchings as guaranteed by Lemma 4.1.9. By counting two ways, $|\mathcal{M}|\frac{n}{2} = \frac{(2r+1)n}{2}p$, which implies that $|\mathcal{M}| = p(2r+1)$. Let $\mathcal{M} = \{M_1, \cdots, M_{p(2r+1)}\}$, and let $w(M_i)$ be the total weight of all edges in M_i. Since $\sum w(M_i) = p\sum_{e\in E(G)} w(e) = pW$, the pigeonhole principle implies that a matching M_j with the smallest weight in the family has weight at most $\frac{1}{2r+1}W$. □

For the proof of the main result, we need the concept of "threads". A *thread* in a graph G is a maximal path in G such that the internal vertices have degree 2 in G.

Theorem 4.1.11. *If $G \in \mathcal{F}_{n,1}$ and $n \geq 10$, then $p(G) \leq \frac{2n-5}{3}$. Equality holds infinitely often for graph $G \in \mathcal{H}'_1$.*

[1] We note that there are at least three different meanings for r-graph in the literature. In [56], for example, r-graph is defined as used here. With this definition, Seymour's r-graph conjecture says that if G is an r-graph, then $\chi'(G) \leq r+1$. In Berge's book [6], "r-graph" is used to mean directed multigraph with multiplicity at most r. In [26], "r-graph" denotes an r-uniform hypergraph.

Proof. Consider $G \in \mathcal{F}_{n,1}$. If G has no balloons or if $n = 10$, then G has a perfect matching and $p(G) = n/2 \leq \frac{2n-5}{3}$. Otherwise, G has a balloon and $n > 10$. For $n > 10$, we proceed by induction. Let e be a cut-edge. Let G_1 and G_2 be the components of $G - e$. Let G_1' and G_2' be the graphs obtained from G by replacing G_2 and G_1, respectively, with B_1. By Lemma 4.1.4, every parity subgraph of G_i' contains e and uses at least two edges in B_1. Such a subgraph can be formed using any parity subgraph of G_i, which has even degree at the endpoint of e in G_i. Hence, $p(G_i') = p(G_i) + 3$ and $p(G) = p(G_1') + p(G_2') - 5$. If neither G_1 nor G_2 is B_1, then G_1' and G_2' are smaller than G. Letting $n_i = |V(G_i')|$, we have $n = n_1 + n_2 - 10$. By applying the induction hypothesis to both G_1' and G_2',

$$p(G) = p(G_1') + p(G_2') - 5 \leq \frac{2n_1 - 5}{3} + \frac{2n_2 - 5}{3} - 5 = \frac{2n - 5}{3}. \tag{4.1}$$

In the remaining case, every cut-edge is incident to a copy of B_1. Let each edge have weight 1. Form G' by deleting all the vertices of all the balloons. If G' is a cycle, then G has a perfect matching and

$$p(G) = \frac{n}{2} < \frac{2n - 5}{3}. \tag{4.2}$$

Otherwise, in G', replace each thread through vertices of degree 2 with a single edge whose weight is the length of the thread. Since the vertices of degree 2 have been suppressed and G' is 2-edge-connected, the resulting weighted graph G'' is a 3-graph by Remark 4.1.8. Thus by applying Lemma 4.1.10, G'' has a perfect matching with at most 1/3 of its total weight. The total weight is $\frac{m-8b}{3}$, where m is the number of edges in G and b is the number of balloons in G. Using Proposition 2.1.2, we have

$$p(G) \leq \ p(G') + 3b \leq \frac{m - 8b}{3} + 3b = \frac{3n - 16b}{6} + 3b = \frac{n}{2} + \frac{b}{3} \leq \frac{n}{2} + \frac{1}{3}(\frac{n+2}{6}) \leq \frac{2n-5}{3}. \tag{4.3}$$

We have proved that $p(G) \leq \frac{2n-5}{3}$ for a connected cubic graph G.

By Proposition 4.1.6, equality holds for graphs in $\mathcal{H}_1$.

$\square$

Note that a 10-vertex connected cubic graph G has $p(G) = 5 = \frac{2*10-5}{3}$ even though G is not in $\mathcal{H}'_1$. However, we show that if G has n vertices for $n \geq 16$ and $p(G) = \frac{2n-5}{3}$, then G must be in $\mathcal{H}'_1$.

Theorem 4.1.12. *If G is a graph in $\mathcal{F}_{n,1}$, then $p(G) = \frac{2n-5}{3}$ if and only if $n = 10$ or $G \in \mathcal{H}'_1$.*

Proof. Since we proved that the condition is sufficient, it suffices to show that if $G \in \mathcal{F}_{n,1}$ and $p(G) = \frac{2n-5}{3}$, then $n = 10$ or $G \in \mathcal{H}'_1$. If $n < 10$, then G has a perfect matching, which implies that $p(G) = \frac{2}{n} > \frac{2n-5}{3}$. Now, assume that $n > 10$. We use induction on n as in the proof of Theorem 4.1.11. To achieve equality in the inequality (4.1), for $i = 1, 2$, G'_i must have $p(G_i) = \frac{2n_i-5}{3}$. Since neither compound obtained by deleting the cut-edge is B_1, we have $|V(G_i)| > 10$. Thus, the induction hypothesis applies, and G'_i is in $\mathcal{H}'_1$, which implies that G must also be in $\mathcal{H}'_1$. In the case, where all cut-edges are incident to balloons, we have three subcases. If deleting the balloons leaves a cycle, then $p(G) = \frac{n}{2} < \frac{2n-5}{3}$. If it leaves a single vertex, then $n = 16$, $b = 3$ and $G \in \mathcal{H}'_1$. If it leaves a graph with minimum degree 2, then $p(G) \leq \frac{5n+1}{9} \leq]frac2n - 53$ (by inequality (4.3), with equality only when $n = 16$. $\square$

4.2 Path Cover Number

If $\mathcal{P}$ is a set of disjoint paths and every vertex in $V(G)$ belongs to exactly one path in $\mathcal{P}$, then we call $\mathcal{P}$ a *path cover* of G. The *path cover number* of G, which we denote by $q(G)$ is the minimum size of such a set. We use $q(G)$ here because we already used $p(G)$ for the parity number of G; in the literature, $p(G)$ is the path covering number of G.

In 1996, Reed [55] proved that if G is a connected n-vertex cubic graph, then $q(G) \leq \lceil \frac{n}{9} \rceil$. Interestingly, equality holds for the graphs in the family $\mathcal{H}_1$ defined in 4.2.1 when $r = 1$.

Example 4.2.1. Let $\mathcal{T}_r$ be the subfamily of $\mathcal{T}'_r$ obtained by requiring all leaves to have the same color in a proper 2-coloring. Let $\mathcal{H}_r$ be the subfamily of $\mathcal{H}'_r$ arising from trees in $\mathcal{T}_r$ by adding balloons at leaves. $\square$

In 2009, Magnant and Martin [41] proved that if G is a connected 4-regular graph with n vertices, then $q(G) \leq \frac{n}{5}$. They used the following lemma.

Lemma 4.2.2. [41] *If G is an r-regular graph, then there is an optimal path cover such that every path in the cover has at least 2 vertices.*

We improve their bound to $\lceil \frac{n}{7} \rceil$.

Lemma 4.2.3. *In a smallest path cover, two consecutive vertices on a path in the cover have together at most two edges incident to endpoints of other paths.*

Proof. If there is a smallest path cover $\mathcal{P}$ such that some consecutive vertices u and v on some path P in $\mathcal{P}$ have together at least three edges incident to endpoints of other paths, then there exist distinct endpoints u' and v' of some paths $P_{u'}$ and $P_{v'}$ such that u and v are adjacent to u' and v', respectively. Now, we have the following two cases.

Case 1: $P_{u'} = P_{v'}$.

Assume that a and b are endpoints of P, which are closer to u and v, respectively. Now, we delete the two paths P and $P_{u'}$ from $\mathcal{P}$ and add the path $P[a,u]uu'P_{u'}v'vP[v,b]$ to $\mathcal{P}$. The new path cover is smaller than $\mathcal{P}$, which is a contradiction.

Case 2: $P_{u'}$ and $P_{v'}$ are different.

If we delete the three paths P, $P_{u'}$ and $P_{v'}$ from $\mathcal{P}$ and add the two paths $P[a,u]uu'P_{u'}$ and $P[b,v]vv'P_{v'}$ to $\mathcal{P}$, then the new path cover is smaller than $\mathcal{P}$, which is a contradiction. □

Theorem 4.2.4. *If G is an n-vertex connected 4-regular graph, then $q(G) \leq \lceil \frac{n}{7} \rceil$.*

Proof. By Lemma 4.2.2, there is an optimal path cover $\mathcal{P}$ using nontrivial paths. Among the paths in $\mathcal{P}$ with i vertices, let $\mathcal{C}_i$ be the family of those whose two endpoints are adjacent in G (completing a cycle), and let $\mathcal{P}_i$ be the family of those whose two endpoints are nonadjacent in G. Let $c_i = |\mathcal{C}_i|$ and $p_i = |\mathcal{P}_i|$. Thus, $\frac{\sum i(c_i+p_i)}{\sum(c_i+p_i)}$ is the average number of vertices among the paths in $\mathcal{P}$. If the average order of paths in $\mathcal{P}$ is at least 7, then $\mathcal{P} \leq \frac{n}{7}$. To prove that the average order is at least 7, we prove

$$\frac{\sum_{i\leq 6} ic_i + \sum ip_i}{\sum_{i\leq 6} c_i + \sum p_i} \geq 7.$$

Let A be the set of edges e in G such that

a) e is not an edge in any path in $\bigcup \mathcal{P}_i$ nor an edge joining the endpoints of a path in $\bigcup \mathcal{C}_i$, and

b) e is incident to an endpoint of a path in $\bigcup \mathcal{P}_i$ or to a vertex of a path in $\bigcup \mathcal{C}_i$.

Orient the edges of A away from end-vertices of paths in $\bigcup \mathcal{P}_i$ and away from vertices in $\bigcup \mathcal{C}_i$. We claim that the number of edges in A going into a path in $\mathcal{P}_i$ minus the number of edges of A going out of the same path is at most

$$\begin{cases} i-8, & \text{if } i \text{ is even,} \\ i-7, & \text{if } i \text{ is odd.} \end{cases}$$

Consider a path P in $\mathcal{P}_i$. The endpoints of P have three departing edges in A and each two consecutive vertices in P have at most two edges coming into the path by Lemma 4.2.3. Thus, if i is even, then we have at most $2\frac{i-2}{2} - 6 = i - 8$ net entering edges. If i is odd, then we have at most $2\frac{i-3}{2} + 2 - 6 = i - 7$ net entering edges.

Since each edge of A must end at some vertex, the sum of the differences in the above claim minus the edges coming from cycles is zero. Because G is a connected even graph, the number of edges in A starting in every element in $\mathcal{C}_6$ or $\mathcal{C}_5$ is at least 2. From a path in $\mathcal{C}_4$ at least one edge must leave each vertex. From a path in $\mathcal{C}_3$, at least two must leave each vertex. Thus, we have

$$\sum p_i\,(i-7) - 2c_6 - 2c_5 - 4c_4 - 6c_3 \geq 0.$$

Thus, $\sum(ip_i) + \sum_{i\leq 6}(ic_i) \geq 7\sum(p_i) + 8c_6 + 7c_5 + 8c_4 + 9c_3 \geq 7(\sum(p_i) + \sum_{i\leq 6}(c_i))$.

□

The graphs in $\mathcal{H}_1$ show that the upper bound $\lceil \frac{n}{9} \rceil$ on the path cover number of graphs in $\mathcal{F}_{n,1}$ cannot be improved. We believe that also the graphs in the family $\mathcal{H}_r$ have the largest path cover number for graphs in $\mathcal{F}_{n,r}$. Similarly, we also conjecture that the graphs in $\mathcal{H}_{r,t}$ and $\mathcal{H}'_{r,t}$ give us the largest path cover number in $\mathcal{F}_{n,r,t}$ and $\mathcal{F}'_{n,r,t}$, where $\mathcal{F}_{n,r,t}$ and $\mathcal{F}'_{n,r,t}$ are the families of $(2t+1)$-edge-connected $(2r+1)$-regular graphs and $2t$-edge-connected $2r$-regular graphs with n vertices, respectively.

Recall that $\mathrm{def}(G) = \max_{S\subseteq V(G)} o(G-S) - |S|$, where $o(G-S)$ is the number of odd components in $G-S$. Next, we determine the path cover number of the graphs in each family mentioned above.

Theorem 4.2.5. *For a graph G, $q(G) \geq \mathrm{def}(G)$.*

Proof. Let S be a set of maximum deficiency in G, and let $G' = G - S$. Note that $q(G') \geq o(G-S)$. Also, $q(G) \geq q(G') - |S|$ because each vertex of S lies on only one path and thus permits combining paths in components of $G - S$ only once. Therefore, $q(G) \geq o(G-S) - |S| = \mathrm{def}(G)$. $\square$

Corollary 4.2.6. *If G is a graph in $\mathcal{H}_r$, then $q(G) = \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1}$.*

Proof. First recall that $\mathrm{def}(G) = \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1}$ by 2.1.2. By 4.2.5, we only need to show that there exists a path cover $\mathcal{P}$ such that $|\mathcal{P}| = \mathrm{def}(G)$.

For a graph $G \in \mathcal{H}_r$, let T be the corresponding tree in $\mathcal{T}_r$. Let X and Y be its partite sets, with Y containing the leaves. Let $S = X$. Now $o(G-S) = |Y|$, since each vertex of Y is an isolated vertex in $G - S$ or is the neck of a copy of B_r that is an odd component of $G - S$. Thus $\mathrm{def}(S) = |Y| - |X|$. Because B_r has a spanning path starting at the neck, $q(G) \leq q(T)$. Since T is an induced subgraph of G, equality holds. We compute $q(T)$ inductively. When $T = K_{1,2r+1}$, we have $\mathrm{def}(S) = 2r$. We can easily find a path cover for T with size $2r$. For larger $G \in \mathcal{H}_r$, let T' with corresponding graph $G' \in \mathcal{H}_r$ be the tree from which T is expanded. In the expansion, $|X|$ increases by $2r$ and $|Y|$ increases by $4r^2$, so $\mathrm{def}(S)$ increases by $4r^2 - 2r$. Comparing T with T', one leaf is lost and $4r^2$ are created; the number of vertices increases by $4r^2 + 2r$. Now, we can add to the path cover for T' a path cover with size $4r^2 - 2r$ for the added vertices. This completes the desired path cover for T. $\square$

Conjecture 4.2.7. *If G is a graph in $\mathcal{F}_{n,r}$, then $q(G) \leq \frac{r}{2r+1}\frac{(2r-1)n+2}{2r^2+2r-1}$.*

Similarly, we conjecture for $(2t+1)$-edge-connected $(2r+1)$-regular graphs with n vertices and for $2t$-edge-connected $2r$-regular graphs with n vertives.

Conjecture 4.2.8. *If G is a graph in $\mathcal{F}_{n,r,t}$, then $q(G) \leq \frac{r-t}{2(r+1)^2+t}\frac{n}{2}$.*

Conjecture 4.2.9. *If G is a graph in $\mathcal{F}'_{n,r,t}$, then $q(G) \leq \frac{r-t}{2r^2+r+t}\frac{n}{2}$.*

If Conjecture 4.2.9 is true, then since the total domination of a k-regular graph G is at least $\frac{1}{k}$, the following conjecture, one of Graffiti.pc conjectures, is true for $k = 4$.

Conjecture 4.2.10. *If G is an k-regular graph, then $2q(G)$ is at most the total domination number of G.*

4.3 Average Connectivity and Average Edge-Connectivity

Connectivity and edge-connectivity of a graph measure the difficulty of breaking the graph apart, but they are very much affected by local aspects like vertex degree. Average connectivity (and analogously, average edge-connectivity) has been introduced to give a more refined measure of the global "amount" of connectivity. In this section, we prove a relationship between the average connectivity and the matching number in all graphs. We also give the best lower bound for the average edge-connectivity over n-vertex connected cubic graphs, and we characterize the graphs where equality holds. In addition, we show that this family has the fewest perfect matchings among cubic graphs that have perfect matchings.

A graph G is *k-connected* if it has more than k vertices and every subgraph obtained by deleting fewer than k vertices is connected. The *connectivity* of G, written $\kappa(G)$, is the maximum k such that G is k-connected. The connectivity of a graph measures how many vertices must be deleted to disconnect the graph. However, since this value is based on a worst-case situation, it does not reflect how well connected the graph is in a global sense. For example, a graph G obtained by adding one edge joining two large complete graphs has the same connectivity as a tree. However, it is much easier to disturb the tree, which is relevant if they both model communication systems.

In 2002, Beineke, Oellermann and Pippert [11] introduced a parameter to measure this difference. *The average connectivity* of a graph G with n vertices, witten $\overline{\kappa}(G)$, is defined to be $\sum_{u,v\in V(G)} \frac{\kappa(u,v)}{\binom{n}{2}}$, where $\kappa(u,v)$ is the minimum number of vertices whose deletion makes v unreachable from u. By Menger's Theorem, $\kappa(u,v)$ is equal to the minimum number of internally disjoint paths joining u and v. Note that $\overline{\kappa}(G) \geq \kappa(G) = \min_{u,v\in V(G)} \kappa(u,v)$.

Regarding average connectivity, several properties are known. The following is one of them.

Theorem 4.3.1. (Dankelmann, Oellermann, 2003) [21] *If G has average degree $\overline{d}$ and n vertices,*

$$\overline{d}\left(\frac{\overline{d}}{n-1}\right) \leq \overline{\kappa}(G) \leq \overline{d}$$

We prove a bound on the average connectivity in terms of matching number. We first introduce the definitions of *M-alternating path* and *M-augmenting path.*

Definition 4.3.2. *Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are missed by M is an M-augmenting path.*

Theorem 4.3.3. *For a connected graph G,*

$$\overline{\kappa}(G) \leq 2\alpha'(G), \tag{4.4}$$

and this is sharp only for complete graphs with an odd number of vertices. In addition, if G is an n-vertex connected bipartite graph, then

$$\overline{\kappa}(G) \leq \left(\frac{9}{8} - \frac{3n-4}{8n(n-8)}\right)\alpha'(G), \tag{4.5}$$

and this is sharp only for the complete bipartite graph $K_{q,3q-2}$, where q is a positive integer.

Proof. First, we show that inequality (4.4) holds for any connected graph G. Let M be a maximum matching in G, and let $m = |M|$. Let $S = V(G) - V(M)$, $s = |S|$, and $n = |V(G)|$. Note that $n = 2m + s$.

If $s \leq 1$, then $m \geq \frac{n-1}{2}$, and the bound holds since $\overline{k}(G) \leq n - 1 \leq 2m$. Thus, we may assume that $s \geq 2$.

For $vv' \in M$, put v and v' into T, T', or R as follows:

If neither v nor v' has a neighbor in S, then put both in T. If v' has a neighbor in S and v does not, then put $v \in T$ and $v' \in T'$. If both have neighbors in S, put them both in R. In this last case, note that if v and v' have distinct neighbors in S, then M is not maximal. Hence each has exactly one neighbor in S, which forms a triangle with them.

We consider three cases to obtain lower bounds on $\kappa(u, v)$ depending on the possible locations of distinct vertices u and v.

Case 1: $u \in S$. First, note that S is independent. Furthermore, if P and P' are distinct internally disjoint u, v-paths, then both of them must visit $V(M) - T$ immediately after u. Since P and P' have no vertex in common, we have $\kappa(u, v) \leq 2m - t$, where $t = |T|$.

Case 2: $u, v \in T'$. Clearly, $\kappa(u, v) \leq n - 1 = 2m + s - 1$

Case 3: $u \in R \cup T$. Recall that every vertex in R has exactly one neighbor in S. For the vertex after u on a u,v-path, at most one vertex of S is available. Thus, if $u \in R \cup T$ and $v \in V(G)$, then there are at most $2m$ candidates to begin such a path.

Thus, we have

$$\overline{\kappa}(G) \leq \frac{(2m-t)\left(\binom{s}{2}+s(n-s)\right)+(2m+s-1)\binom{t'}{2}+2m\left(\binom{n}{2}-\binom{s}{2}-\binom{t'}{2}-s(n-s)\right)}{\binom{n}{2}}$$

$$\leq \frac{(2m-t)\binom{s}{2}+(2m+s-1)\binom{t'}{2}+(2m-t)ts+2m(\binom{n}{2}-\binom{s}{2}-\binom{t'}{2}-ts)}{\binom{n}{2}}$$

$$= 2m + \frac{(s-1)\binom{t'}{2}-t\binom{s}{2}-t^2 s}{\binom{n}{2}} = 2m - t\frac{s^2+3ts+t-1}{n(n-1)} \leq 2m. \tag{4.6}$$

The last inequality of (4.6) holds because when $t \geq 1$, $s^2+3ts+t-1 \geq 0$ and when $t=0$, we have $t\frac{s^2+3ts+t-1}{n(n-1)} = 0$.

To have equality in the last inequality of (4.6), we need to have $t=0$ or $t=1$.

When $t=1$, equality requires $s=0$, which implies that M is a perfect matching. Thus $2m = \frac{n}{2}$. In this case, $\kappa(G) \leq n-1 < n = 2m$, which implies that we cannot have equality in (4.6).

If $t=0$ and $s \geq 2$, then we have a bigger matching than M, since every vertex in R has exactly one neighbor in S and G is connected. Thus, when $t=1$, equality requires $s=1$, which implies $2m = n-1$. If $\kappa(G) = n-1$, then $G = K_n$. Thus, equality holds only when G is a complete graph with an odd number of vertices.

To prove that inequality (4.5) holds, we consider an n-vertex connected bipartite graph G with partite sets A and B. Let M be a maximum matching in G. Let $m = |M|$, let $A_1 = A - V(M)$, and let $B_1 = B - V(M)$. Let B_2 be all vertices in B that are reachable by an M-alternating path from a vertex in A_1, and let A_2 be all vertices in A that are reachable by an M-alternating path from a vertex in B_1. Note that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$ and there are no edges of M joining A_2 and B_2; otherwise we have a bigger matching than M by making a M-augmenting path, which is a contradiction. Let $A_3 = A - (A_1 \cup A_2)$ and $B_3 = B - (B_1 \cup B_2)$. Let $|A_i| = a_i$ and $|B_i| = b_i$ for $i = 1, 2, 3$.

We consider five cases to obtain lower bounds on $\kappa(u,v)$ depending on the possible locations of distinct vertices u and v.

Case 1: $u, v \in A_2$

Since every u, v-path must pass through a vertex in B, we have $k(u, v) \leq b = m + b_1$.

Case 2: $u, v \in B_2$

Since every u, v-path must pass through a vertex in A, we have $k(u, v) \leq a = m + a_1$.

Case 3: $u \in A$ and $v \in (A - A_2)$.

Since every u, v-path must pass through at least one vertex in $B \cap V(M)$, we have $k(u, v) \leq m$.

Case 4: $u \in B$ and $v \in (B - B_2)$.

Since every u, v-path must pass through at least one vertex in $A \cap V(M)$, we have $k(u, v) \leq m$.

Case 5: $u \in A - A_2$ and $v \in B - B_2$

Every u, v-path must pass through at least two vertices in M except the path of length one uv, which implies that $k(u, v) \leq m$.

Thus, we have

$$\overline{k}(G) \leq \frac{m\binom{n}{2} + a_1\binom{b_2}{2} + b_1\binom{a_2}{2}}{\binom{n}{2}} = m + \frac{a_1\binom{b_2}{2} + b_1\binom{a_2}{2}}{\binom{n}{2}} \leq m + \frac{(a_1 + b_1)\binom{b_2+a_2}{2}}{\binom{n}{2}}. \tag{4.7}$$

Since no edge of M joins A_2 to B_2, all vertices of A_2 match into B_3 under M. Thus, we have $a_2 \leq b_3$. Similarly, we have $b_2 \leq a_3$. Thus, we have $(a_1 + b_1) + 2(a_2 + b_2) \leq n$ and $(a_2 + b_2) \leq m$. Since $n-2 \geq (a_1+b_1)+2(a_2+b_2-1) \geq 2\sqrt{2(a_1 + b_1)(a_2 + b_2 - 1)}$, we have $(a_1+b_1)(a_2+b_2-1) \leq \frac{(n-2)^2}{8}$. Thus, we have

$$\begin{aligned} \overline{k}(G) &\leq m + \frac{(a_1 + b_1)\binom{b_2+a_2}{2}}{\binom{n}{2}} \leq m + \frac{(a_1 + b_1)(a_2 + b_2 - 1)}{n(n-1)}(a_2 + b_2) \\ &\leq m + \frac{(n-2)^2}{8n(n-1)}m = \frac{9}{8}m - \frac{3n-4}{8n^2 - 8n}m. \end{aligned}$$

To have equality in the last inequality of (4.7), $a_1 = 0$, $b_2 = 0$ or $b_1 = 0$, $a_2 = 0$. Equality holds for $K_{q,3q-2}$, since $\alpha'(K_{q,3q-2}) = q$ and $\overline{\kappa}(K_{q,3q-2}) = \frac{\binom{4q-2}{2}q + \binom{q}{2}(2q-3)}{\binom{4q-2}{2}} = q + \frac{q(q-1)^2}{(2q-1)(4q-3)} = \frac{9}{8}q - \frac{3(4q-2)-4}{8(4q-2)(4q-3)}q$.

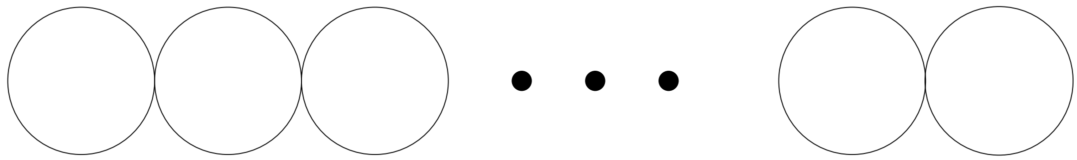

Figure 4.1: A graph G_1 with $\overline{\kappa}(G_1) = 1 + O(\frac{q}{s})$ and $\overline{\kappa'}(G_1) = q - 1$

$\square$

Next, we introduce the concept for edges analogous to average connectivity. A graph G is *k-edge-connected* if every subgraph obtained by deleting fewer than k edges is connected; the *edge-connectivity* of G, written $\kappa'(G)$, is the maximum k such that G is k-edge-connected. *The average edge-connectivity* of a graph G with n vertices, witten $\overline{\kappa}'(G)$, is defined to be $\sum_{u,v \in V(G)} \kappa'(u,v) / \binom{n}{2}$, where $\kappa'(u,v)$ is the minimum number of edges whose deletion makes v unreachable from u, which is same as the number of edge-disjoint pathes between u, v. Note that $\overline{\kappa}'(G) \geq \kappa'(G) = \min_{u,v \in V(G)} \kappa'(u,v)$.

This new parameter shares certain properties with the average connectivity. Even if we replace $\overline{\kappa(G)}$ in Theorem 4.3.1 and Theorem 4.3.3 by $\overline{\kappa'(G)}$, then the inequalities hold.

Theorem 4.3.4. *If G has average degree d, and $|V(G)| = n$,*

$$\frac{d^2}{n-1} \leq \overline{k'}(G) \leq d$$

The proof is the same as the proof of Dankelmann and Oellermann.

Theorem 4.3.5. *If G has matching number α',*

$$\overline{k'}(G) \leq 2\alpha'(G)$$

If G is bipartite,

$$\overline{k'}(G) \leq \frac{9}{8}\alpha'(G) - \frac{3n-4}{8n^2 - 8n}\alpha'(G)$$

The proof is the same as in Theorem 4.3.3 if we look at the second vertex in sets of edge-disjoint paths.

However, consider the graph G_1 in figure 1, which is the graph obtained from P_{s+1} by replacing an edge with a copy of K_q. Two successive copies of K_q share one vertex. The total number of vertices is $1+s(q-1)$. Since $\overline{\kappa}(G_1) = 1+O(\frac{q}{s})$ and $\overline{\kappa'}(G_1) = a-1$, we have a big difference between the average connecitivity and the average edge-connectivity of this graph.

In order to analyze average edge-connectivity of a regualr graph, we define several notions. If a graph G has a cut-edge, then we get components after we delete all cut-edges of G. We define an *i-balloon* to be such a component incident to i cut-edges. Note that 1-balloon is a balloon and for any $i \geq 1$, an i-edge-ballon is a maximal 2-edge-connected subgraph of G except when it is a single vertex, and the resulting graph obtained by schrinking each i-balloon to a single vertex is a tree. For a cubic graph, its smallest 1-balloon is the smallest possible balloon in a cubic graph, which is B_1. The smallest 2-ballon is $K_4 - e$. We denote the smallest i-edge-balloon in a cubic graph by B_i. Now we compute the average edge-connectivity of several cubic graphs with n vertices less than 10. Before doing it, we first give a lemma.

Lemma 4.3.6. *If G has a vertex subset S in $V(G)$ such that $|[S,\overline{S}]| < \delta(G)$, then $|S| \geq \delta(G)+1$. Furthermore, if G is a $(2r+1)$-regular graph and S is a vertex subset in $V(G)$ such that $|[S,\overline{S}]| < 2r+1$, then $|[S,\overline{S}]| \equiv |S| \mod 2$*

By the Degree-Sum Formula, we consider only even number of vertices. If $n = 4$, then it is K_4. Since edge-connecitivy of K_4 is 3, we have $\kappa'(K_4) = 3\binom{4}{2}$, which is greater than $\binom{4}{2} + \frac{7\times 4+58}{4}$. If $n = 6$, then $\kappa'(G) = 3$ since if its edge-connectivity is less than 3, then it has to have at least 8 vertices. Thus, the average edge-connectivity of a cubic graph with 6 vertices, $3\binom{6}{2}$, is greater than $\binom{6}{2} + \frac{7\times 6+58}{4}$. If $n = 8$, then its edge-connectivity is at least 2 since if its edge-connecitivyt is equal to 1, then it has to have at least 10 vertices. It its edge-connectivity is equal to 2, then it is the graph obtained by adding two edges between two B_1's. Its edge-connectivity is $2\binom{8}{2}+2$, and note that $3\binom{8}{2} \geq 2\binom{8}{2}+2$. Thus, The average edge-connectivity of a cubic graph with 8 vertices is greater than $\binom{8}{2} + \frac{7\times 8+58}{4}$. Now, we prove that every cubic graph other than K_4 satisfies the bound in the following theorem.

Theorem 4.3.7. *If G is a connected cubic graph G with n vertices, which is not K_4, then*

$$\kappa'(G)\binom{n}{2} \geq \binom{n}{2} + \frac{7n+58}{4}.$$

Proof. Consider a minimal counterexample G with n vertices.

Claim 1: The edge-connectivity of G is 1. If not, then $\kappa'(G)\binom{n}{2} \geq 2\binom{n}{2} \geq \binom{n}{2} + \frac{7n+58}{4}$ for $n \geq 10$. In the above, we showed that a cubic graph with vertices less than 10 satisfies the bound when it is not K_4.

claim 2: Every 1-balloon of G is B_1. If D_1 is an 1-balloon of G with $D_1 \neq B_1$ and $|V(D_1)| = 5+a$, then Degree-Sum formula guarantees that a is an even positive integer, which imples that $a \geq 2$. Let G' be the graph obtained from G by replacing D with B_1. Note that G' is cubic. In addition, since G' also has a cut-edge, $10 \leq n-a = |V(G')| \leq |V(G)|$. Since G is larger graph than G', which is not K_4, by the hopothesis of G, we have $K'(G')\binom{n-a}{2} \geq \binom{n-a}{2} + \frac{7}{4}(n-a) + \frac{29}{2}$. By the construction of G' and the fact that B is 2-edge-connected,

$$K'(G)\binom{n}{2} = K'(G')\binom{n-a}{2} - K'(B_1)\binom{5}{2} - 5(n-a-5) + K'(B)\binom{5+a}{2} + (5+a)(n-a-5)$$

$$\geq \binom{n-a}{2} + \frac{7}{4}(n-a) + \frac{29}{2} - 26 - 5(n-a-5) + 2\binom{5+a}{2} + (5+a)(n-a-5)$$

$$= \binom{n}{2} + \frac{a^2+a-2an}{2} + \frac{7}{4}n - \frac{7a}{4} + \frac{29}{2} - 26 + (5+a)(5+a-1) + a(n-a-5)$$

$$= \binom{n}{2} + \frac{7}{4}n + \frac{2a^2+11a+34}{4} > \binom{n}{2} + \frac{7}{4}n + \frac{29}{2}$$

for $a \geq 2$, which is a contradiction to the assumption that G is a counterexample.

Claim 3: Every 2-balloon of G is B_1'. If D_2 is an 2-balloon of G with $D_2 \neq B_1'$ and $|D_2| = 4+a$, then then Degree-Sum formula guarantees that a is an even positive integer, which imples that $a \geq 2$. Let G' be the graph obtained from G by replacing D_2 with B_1' in order that G' is a cubic graph. Thus, G' has $n-a$ vertices for $a \geq 2$, and by the hypothesis that G is a minimal counterexample, $K'(G')\binom{n-a}{2} \geq \binom{n-a}{2} + \frac{7}{4}(n-a) + \frac{29}{2}$. By the construction of G', and the fact that

B' is 2-edge-connected, we have

$$K'(G)\binom{n}{2} = K'(G')\binom{n-a}{2} - K'(B_1')\binom{4}{2} - 4(n-a-4) + K'(B)\binom{4+a}{2} + (4+a)(n-a-4)$$

$$\geq \binom{n-a}{2} + \frac{7}{4}(n-a) + \frac{29}{2} - 13 - 4(n-a-4) + 2\binom{4+a}{2} + (4+a)(n-a-4)$$

$$= \binom{n}{2} + \frac{a^2+a-2an}{2} + \frac{7}{4}n - \frac{7a}{4} + \frac{29}{2} - 13 + a(n-a-4) + (4+a)(4+a-1)$$

$$= \binom{n}{2} + \frac{7}{4}n + \frac{29}{2} + \frac{2a^2+7a+48}{4} > \binom{n}{2} + \frac{7}{4}n + \frac{29}{2}$$

for $a \geq 2$, which is a contradiction to the assumption that G is a counterexample.

Claim 4: G has no k-balloons for $k \geq 3$. Assume that G has a k-balloon for $k \geq 3$. Since G contains B_1 as an induced subgraph by *Claim 1*, choose a k-balloon D_k for $k \geq 3$ which is closest to B. D_k is incident to B_1's or B_1 by the choice of D_k. If $k \geq 4$, then $|V(D_k)| \geq k$ since each vertex in $V(D_k)$ is incident to at most one cut-edge. Thus, we can assume that $|V(D_k)| = k + a$ with $a \geq 0$. Suppose that there are m B_4s between B and D. Let G' be the graph obtained from G by deleting all B_4s between D_k and B_1, deleting B_1, and replacing D_k with C_{k-1} and attching each cut-edge except one between D and B to each vertex in C_{k-1}. Note that G' has $n - a - 4m - 6$ vertices. Thus, we have $K'(G')\binom{n-a-4m-6}{2} \geq \binom{n-a-4m-6}{2} + \frac{7}{4}(n-a-4m-6) + \frac{29}{2}$. The construction of G' guarantees that

$$K'(G)\binom{n}{2} = K'(G')\binom{n-a-4m-6}{2} - K'(C_{k-1})\binom{k-1}{2}$$

$$-(k-1)(n-a-4m-k-5) + \binom{4m+5+a+k}{2} + (K'(D)-1)\binom{k+a}{2}$$

$$+m(K'(B_4)-1)\binom{4}{2} + (K'(B_5)-1)\binom{5}{2} + (4m+5+a+k)(n-4m-a-k-5)$$

$$\geq \binom{n-a-4m-6}{2} + \frac{7}{4}(n-a-4m-6) + \frac{29}{2} - 2\binom{k-1}{2} - (k-1)(n-a-4m-k-4)$$

$$+\binom{4m+5+a+k}{2} + \binom{k+a}{2} + 7m + 16 + (4m+5+a+k)(n-4m-a-k-5)$$

$$= \binom{n}{2} + \frac{7}{4}n + \frac{1}{2}a^2 - \frac{9}{4}a + 19 + k + ka = \binom{n}{2} + \frac{7}{4}n + \frac{29}{2} + \frac{1}{2}((a - \frac{9}{4})^2 + \frac{63}{16}) + k + ka \geq \binom{n}{2} + \frac{7}{4}n + \frac{29}{2}$$

, which is a contradiction to the hypothesis that G is a counterexample.

Thus, we can assume that k is equal to exactly 3. Let $|V(D_3)| = a$. Note that $a \geq 1$. Assume that there are m B_4s between B and D. Let G' be the graph obtained from G by deleting all B_4s between D_3 and B, deleting B, replacing D with B_4, and attaching each of two remaining cutedges to vertices of B_4 with degree 2. Note that G' has $n - a - 4m - 1$ vertices. Since G' is smaller than G, we have $K'(G')\binom{n-a-4m-a}{2} \geq \binom{n-a-4m-1}{2} + \frac{7}{4}(n - a - 4m - 1) + \frac{29}{2}$. By the construction of G', we have

$$K'(G)\binom{n}{2} = K'(G')\binom{n - a - 4m - 1}{2} - K'(B_4)\binom{4}{2} - (4)(n - a - 4m - 5)$$

$$+ \binom{a + 4m + 5}{2} + (\overline{k'}(D) - 1)\binom{a}{2} + m(\overline{k'}(B_4) - 1)\binom{4}{2} + (\overline{k'}(B_5) - 1)\binom{5}{2} + (a + 4m + 5)(n - a - 4m - 5)$$

$$\geq \binom{n - a - 4m - 1}{2} + \frac{7}{4}(n - a - 4m - 1) + \frac{29}{2} - 13 - 4(n - a - 4m - 5) + \binom{a + 4m + 5}{2} + \binom{a}{2}$$

$$+7m + 16 + (a + 4m + 5)(n - a - 4m - 5) = \binom{n}{2} + \frac{7}{4}n + \frac{1}{2}a^2 - \frac{9}{4}a + \frac{87}{4} + 4k$$

$$= \binom{n}{2} + \frac{7}{4}n + \frac{1}{2}(a - \frac{9}{4})^2 + \frac{615}{32} + 4k > \binom{n}{2} + \frac{7}{4}n + \frac{29}{2}$$

,which is a contradiction to the assumption that G is a counterexample. Therefore, G contains no k-balloons for $k \geq 3$. By the above claims, the minimal counter example should be a graph consists of only B_4 and B_5. After contracting each edge-block of G, we should get tree with maximum degree 2, which is a path. Thus G should be two B_5 on the endvertices and B_4s are attached each other. But in that case, $\overline{k'}(G) = \binom{n}{2} + \frac{7}{4}n + \frac{29}{2}$, which satisfy the proposition.

Thus, it contradicts the assumption that G is a counterexample. And the inequality is sharp only for $n \equiv 2 \ mod \ 4$ by the above example.

□

Figure 4.2 describes an infinite family of graphs for which equality holds in Theorem 1.3.4. We make the following conjecture for $(2r + 1)$-regular graphs.

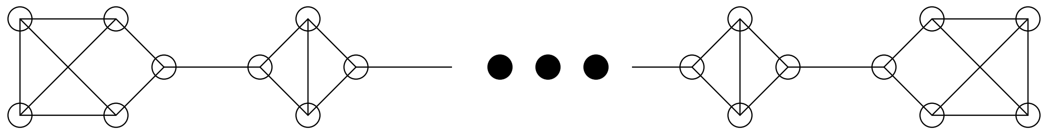

Figure 4.2: A graph for which equality in Theorem 1.3.4 holds

Conjecture 4.3.8. *If G is a connected $(2r+1)$-regular graph G with n vertices, then*

$$K'(G)\binom{n}{2} \geq \min\{2\binom{n}{2}, \binom{n}{2} + \frac{(r-2)(r^2+2r-1)}{2(r+1)}n + \frac{r^3+4r^2+r-8}{r+1}\}.$$

If the above conjecture holds, then we know that it is sharp for infinitely many n. Let $A = B_r$ and $B = K_{2r+2} - e$. Consider a path P with length at least 1. Replace both end-vertices of P by A and the other vertices of P by B. We define a $(2r+1)$-chain to be the resulting graph.

The graphs when equality in Theorem 4.3.7 holds are also helpful to find a lower bound for the number of perfect matchings in cubic graphs.

We denote the number of perfect matchings in G by $pm(G)$.

Theorem 4.3.9. (Edmonds, Lov'asz and Pulleyblank [24]; Naddef [42]) *If G is an n-vertex connected cubic graph without cut-edges, then $pm(G) \geq \frac{n}{4} + 2$.*

We will use Plésnik's Theorem, which states that if G' is the graph obtained from a $(k-1)$-edge-connected k-regular multigraph G by deleting at most $k-1$ edges in G, then G' has a perfect matching.

Lemma 4.3.10. *If B is a balloon with the nect v in a cubic graph, then there are at least two near perfect matchings not using v.*

Proof. Let x and y be the two vertices adjacent to v in B. Let B' be the resulting graph obtained from B by adding an edge between x and y after deleting the vertex v. Note that B' is a cubic multigraph without cut-edges. By Plésnik's Theorem, there are at least two perfect matchings not using the added edge, which implies that there are at least two near perfect matchings in B. □

Theorem 4.3.11. *Every n-vertex connected cubic graph with a perfect matching except for K_4 has at least four perfect matchings. In addition, equality holds for all 3-chains.*

Proof. Assume that G is a connectec cubic graph with n vertices other than K_4. Note that $n \geq 6$. If G has no cut-edges, then by Theorem 4.3.9, G has at least four perfect matchings.

Now, assume that G has a cut-edge. Hence we have at least two balloons, by 2.1.6. By Lemma 4.3.10, each balloon has at least two near-perfect matchings not using its neck, where the neck of a balloon is the vertex with degree 2. Since there are are at least two balloons, we have at least four perfect matchings.

Consider 3-chain G. There are exactly two near perfect matchings in each copy of B_1 in G and since every perfect matching in G has to use all cut-edges in G, we have only one choice in each copy of $K_4 - e$. Thus, we have exactly four perfect matchings in G. $\square$

Chapter 5

r-dynamic Coloring of Graphs

An r-dynamic proper k-coloring of a graph G is a proper k-coloring of G such that every vertex in $V(G)$ has neighbors in at least $\min\{d(v), r\}$ different color classes. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the least k such that G has an r-dynamic proper k-coloring. Our main result in this chapter is that if G is a k-regular graph and $k \geq 7r \ln r$, then $\chi_r(G) \leq r\chi(G)$, where $\chi(G)$ is the chromatic number of G. In addition, we study the 2-dynamic chromatic number of a graph and the r-dynamic chromatic number of the cartesian product of two graphs.

5.1 Introduction

A teacher makes the following assignment: Each student must choose a country to study and explain to his or her friends. Each student with at least r friends must hear from friends about r different countries. A student with fewer friends must hear about different countries from all friends. In both cases, no two friends can study the same country. The students can plan together who will study which country. How many countries are needed? This problem models *r-dynamic coloring* of graphs.

An *r-dynamic proper k-coloring* of a graph G is a proper coloring c from $V(G)$ to a set S of k colors such that $|c(N(v))| \geq \min\{r, d(v)\}$ for each vertex v in $V(G)$, where $c(S) = \{c(v)\colon\ v \in S\}$ for a vertex subset S. The *r-dynamic chromatic number of a graph G*, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. Thus, $\chi_r(G)$ is the number of countries the students need. (Put a history about conditional coloring.)

The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in the papers [40], [1], [2], and [37]. Montgomery introduced the notion of *dynamic chromatic number* in his dissertation [40]. He conjectured that if G is a regular graph, then $\chi_2(G) \leq \chi(G) + 2$, which is still open. In [1], Akbari, Ghanbari, and Jahanbekam proved that the conjecture is true for bipartite regular graphs. They also studied the dynamic chromatic number of the cartesian product of two graphs in [2] and the list dynamic chromatic number of general graphs in [3]. In 2003, Lai, Montgomery, and Poon [37] proved that $\chi_2(G) \leq \Delta(G) + 1$ except for C_5.

Here are some simple examples to illustrate the definiton of r-dynamic coloring when $r = 2$. The 2-dynamic chromatic number of P_n is just equal to its chromatic number when $n \leq 2$. If $n \geq 3$, then $1, 2, 3, 1, 2, 3, \ldots,$ is a 2-dynamic 3-coloring of P_n. Since three consecutive vertices of P_n must be colored differently, $\chi_2(P_n) = 3$ for $n \geq 3$. Similarly, $\chi_2(C_n)$ is equal to 3 if n is a multiple of 3, equal to 5 if $n = 5$, and equal to 4 otherwise.

Note that $\chi_r(G) \leq \chi_{r+1}(G)$, since an $(r+1)$-dynamic coloring of G is an r-dynamic coloring of G, by definition. Since each vertex of a path or a cycle has degree at most 2, we have $\chi_r(P_n) = \chi_2(P_n)$ and $\chi_r(C_n) = \chi_2(C_n)$ for $r \geq 2$. However, if $\Delta(G) > 2$, then $\chi_2(G)$ and $\chi_3(G)$ may be different. For example, if P is the Petersen graph, then $\chi_2(P) = 4$ and $\chi_3(P) = 10$.

The Kneser graph $K(n, k)$ is the graph with the vertex set $\binom{[n]}{k}$ in which u is adjacent to v if and only if $u \cap v = \emptyset$. The Petersen graph P is the Kneser graph $K(5, 2)$. Although we know $\chi_r(P)$ for every r, we do not know $\chi_r(K(n, k))$ in general. When $r = 2$, we know that $\chi_2(K(n, k)) \leq \chi(K(n, k)) + 2 = n - 2k + 4$, which gives support to the conjecture that $\chi_2(G) \leq \chi(G) + 2$ when G is regular. Also, it is easy to show that $\chi_2(K(n, k)) = n - 2k + 2 = \chi(K(n, k))$ for $n \geq 3k$. For $2k \leq n < 3k$, we believe that $\chi_2(K(n, k)) = n - 2k + 3 = \chi(K(n, k)) + 1$. However, we do not even guess for $\chi_r(K(n, k))$ when $r \geq 3$.

The following observations are immediate from the definition.

Observation 5.1.1. $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$

Observation 5.1.2. *If* $\Delta(G) \leq r$, *then* $\chi_r(G) = \chi_{\Delta(G)}(G)$.

Observation 5.1.1 holds with equality for trees.

Theorem 5.1.3. *For a tree* T, $\chi_r(T) = \min\{\Delta(T), r\} + 1$

Proof. If we take one vertex as a root and iteratively color children of each vertex greedily with colors different from the parent, then the specified number of colors suffices. □

5.2 $\chi(G)$ and $\chi_r(G)$

There are many upper bounds and lower bounds for the chromatic number of a graph in terms of graph parameters. For example, $\chi(G) \leq \Delta(G) + 1$. The same bound holds for the 2-dynamic chromatic number of G, except for C_5 [37]. Some graphs achieve equality, like Petersen graph or C_n when n is a multiple of 3. However, we still cannot characterize when equality holds.

Since $\chi_2(G) \geq \chi(G)$, proving that an upper bound on $\chi(G)$ is also an upper bound on $\chi_2(G)$ is a stronger result. Another upper bound on the chromatic number is $\chi(G) \leq 1 + l(G)$, where $l(G)$ is the length of a longest path in G. Also, $\chi_2(G) \leq l(G) + 1$ [40]. However, the fact that $\chi(G) \leq 1 + l(D)$ for any orientation D of G is not true for $\chi_2(G)$. Here are counterexamples. Let $A = \{1, 2, \cdots, n\}$ and $B = \binom{n}{[r]}$. For $a \in A$ and an r-subset $b \in B$, if $a \in b$, then include the edge ab. Let H be the resulting graph. Note that for $a \in A$, $d(a) = \binom{n-1}{r-1}$ and for $b \in B$, $d(b) = r$. Since H is bipartite, $\chi(H) = 2$. Furthermore, $\chi_r(H) = n$ since no r-dynamic coloring gives the same color to two vertices in A, but when vertices of A have distinct colors, it is easy to complete an r-dynamic. This example shows that the gap between $\chi(G)$ and $\chi_r(G)$ may be unbounded. It also shows that $\chi_r(G)$ cannot be bounded by $1 + l(D)$ when D is an orientation of G, since if we orient each edge from A to B, then the length of a longest path in the resulting orientation is just 2 and $\chi_r(G)$ is arbitrary large.

In 2010, Akbari et al. [1] proved that $\chi_2(G) \leq 2\chi(G)$ for a regular graph G. More generally, we prove $\chi_r(G) \leq r\chi(G)$ for large enough r. In fact, it is not true for $r = 3$, since for the Petersen graph P, we have $\chi_3(P) = 10$ and $\chi(P) = 3$.

For k-regular graphs, we can use a random r-coloring of the vertices to show that $\chi_r(G) \leq r\chi(G)$ when k is sufficiently large in terms of r. We need the probability that some r neighbors of each vertex have distinct colors. The *Stirling number* $S(k, r)$ of the second kind is the number of partitions of $[k]$ into r (nonempty) unlabeled blocks.

Lemma 5.2.1. *Let H be a k-uniform hypergraph, and fix r with $r \leq k$. Color the vertices of H from a set of r colors, with each vertex receiving each color with probability $\frac{1}{r}$ independently. For each $e \in E(H)$, the probability that e receives fewer than r colors is $\frac{r!S(k,r)}{r^k}$.*

Proof. The colorings of e using all r colors correspond to partitions of k elements into r nonempty labeled blocks. By the definition of the Stirling number, there are $r!S(k,r)$ such colorings. They are equally likely. $\square$

Lemma 5.2.2. (Symmetric Local Lemma) *Let $A_1, ..., A_n$ be events such that each is mutually independent of some set of all but $d-1$ of the other events, and suppose that $P(A_i) \leq p$ for all i. If $\mathrm{e}pd < 1$, where $\mathrm{e} = 2.71828\cdots$, then $P(\cap \overline{A_i}) > 0$.*

Lemma 5.2.3. *If H is a k-uniform hypergraph and $\mathrm{e}p(k(\Delta(H)-1)+1) < 1$, where $p = 1 - \frac{r!S(k,r)}{r^k}$, then there is an r-coloring of $V(H)$ such that every edge in $E(H)$ has r colors on it.*

Proof. Color the vertices of H independently and uniformly at random from a set of r colors. For any $e \in E(H)$, let A_e be the event that e has at most $r-1$ colors. The event A_e is determined by choices on the vertices of e, so A_e is mutually independent of all events for edges that do not intersect e. These include all but at most $k(\Delta(H)-1)+1$ events. By Lemma 5.2.1, $P(A_e) = 1 - \frac{r!S(k,r)}{r^k}$. By Lemma 5.2.2, since $\mathrm{e}p(k(\Delta(H)-1)+1) < 1$ by hypothesis, there exists an outcome of the coloring in which each edge of H has r colors. $\square$

Theorem 5.2.4. *If G is a k-regular graph and $\mathrm{e}p(k(k-1)+1) < 1$, where $p = 1 - \frac{r!S(k,r)}{r^k}$, then $\chi_r(G) \leq r\chi(G)$.*

Proof. Define a hypergraph H such that $V(H) = V(G)$ and $E(H) = \{N(v) : v \in V(G)\}$. Thus, H is a k-uniform hypergraph with $\Delta(H) = k$. By Lemma 5.2.3, since $\mathrm{e}p(k(k-1)+1) < 1$ by hypothesis, there is an r-coloring of $V(H)$ such that every edge in $E(H)$ has r colors on it. Let c_1 be such an r-coloring of H, and let c_2 be a proper $\chi(G)$-coloring of G. If we let $c(v) = (c_1(v), c_2(v))$ for $v \in V(G)$, then c is an r-dynamic $r\chi(G)$-coloring of G, which implies that $\chi_r(G) \leq r\chi(G)$. $\square$

Now, we may wonder how large k needs to be in terms of r so that the inequality $\mathrm{e}p(k(k-1)+1) < 1$ in Theorem 5.2.4 holds.

Lemma 5.2.5. *If p is the probability that a given edge of a k-uniform hypergraph H receives fewer than r colors in a random r-coloring of $V(H)$, then $p \leq \frac{(r-1)^k}{r^{k-1}}$.*

Proof. By Lemma 5.2.1, $p = 1 - \frac{r!S(k,r)}{r^k}$. A standard elementary computation by Includsion-Exclusion yields $S(k,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^k$. Thus, $p \leq \frac{r(r-1)^k}{r^k} \leq \frac{(r-1)^k}{r^{k-1}}$. □

Corollary 5.2.6. *If G is a k-regular graph with $k \geq 7r\ln r$ and $r \geq 2$, then $\chi_r(G) \leq r\chi(G)$.*

Proof. When $r = 2$, we have $\mathrm{e}\frac{(r-1)^k}{r^{k-1}}(k(k-1)+1) = \mathrm{e}\frac{1}{2^{k-1}}(k^2-k+1) < \frac{1}{2^{k-3}}k^2 < 1$ for $k \geq 7 * 2\ln 2 > 9.7$ since $2^{k-3} > k^2$ for $k > 9.7$. When $r = 3$, we have $\mathrm{e}\frac{2^k}{3^{k-1}}(k^2-k+1) < \mathrm{e}\frac{2^k}{3^{k-1}}k^2 < 1$ for $k \geq 7 * 3\ln 3 > 23$ since $(\frac{3}{2})^{k-1} > 2\mathrm{e}k^2$ for $k \geq 23$. Let $f(r,k) = \ln(\mathrm{e}\frac{(r-1)^k}{r^{k-1}}k^2)$. Since $\mathrm{e}\frac{(r-1)^k}{r^{k-1}}(k(k-1)+1) < \mathrm{e}^{f(r,k)}$, we only need to show $f(r,k) < 0$ for $r \geq 4$. By the familiar inequality $1 - x < \mathrm{e}^{-x}$, we have $\ln(1-\frac{1}{r}) < -\frac{1}{r}$. By letting $g(r) = f(r, 7r\ln(r))$, we have

$$g(r) = 1 + 2\ln 7 + 3\ln r + 2\ln(\ln r) + 7r\ln r\ln(1-\frac{1}{r})$$

$$< 1 + 2\ln 7 + 3\ln r + 2\ln(\ln r) + 7r\ln r(-\frac{1}{r})$$

$$= 1 + 2\ln 7 + 2\ln(\ln r) - 4\ln r.$$

By setting $h(r)$ to be the last expression, we have $g(r) < h(r)$. Since $h(4) < 0$ and $h'(r) = \frac{2}{rlnr} - \frac{4}{r} = \frac{2}{r}\frac{1}{\ln r} - 2 < 0$, we have $h(r) < 0$ for $r \geq 4$. Since $g(r) < h(r)$, $g(r) < 0$ for $r \geq 4$. Thus, $g(r) < 0$ for $r \geq 4$. Now, with $k = 7r\ln r$, we have that $f(r,k)$ is negative for $r \geq 4$. If we differentiate $f(r,k)$ in terms of k, then

$$\frac{d}{dk}f(r,k) = \frac{2}{k} + \ln(r-1) - \ln(r) < \frac{1}{4r\ln r} + \ln(1-\frac{1}{r})$$

$$< \frac{1}{4r\ln r} + (-\frac{1}{r}) = \frac{1-4\ln(r)}{4r\ln r} < 0$$

for $k \geq 7r\ln r$ and $r \geq 4$. Thus, $f(r,k)$ decreases as k increases for $k \geq 6r\ln r$. Since we showed that $f(r, 7r\ln r) < 0$, also $f(r,k) < 0$ for $k \geq 7r\ln r$, so $\mathrm{e}\frac{(r-1)^k}{r^k}(k(k-1)+1) < \mathrm{e}^{f(r,k)} < 1$ for $r \geq 2$. □

When r is small, we can find better bounds for k than $7r\ln r$ for the application of Corollary 5.2.6. For example, $k = 8, 18, 30, 43, 56$ are enough for $r = 2, 3, 4, 5, 6$, respectively. Furthermore, the Theorem 5.2.4 guarantees that if r is large enough, and G is k-regular, then $\chi_r(G) \leq r(k+1)$, since $\chi(G) \leq k+1$.

Now, we propose the following conjecture:

Conjecture 5.2.7. *For fixed r, it is ture for every graph G with sufficiently large maximum degree that $\chi_r(G) \leq (r-1)(\Delta(G)+1)$ except for finitely many graphs.*

By [37], we know that it is true except for C_5 when $r = 2$. In general, we do not even know whether $\chi_r(G) \leq r(\Delta(G)+1)$ except for finitely many graphs. Note that since the 3-dynamic chromatic number of the Petersen graph is 10, it is not true that $\chi_r(G) \leq r\chi(G)$ for every regular graph and every r.

5.3 2-Dynamic Coloring

The case $r = 2$ of r-dynamic coloring was previously studied under the name "dynamic coloring". In this section, we study some properties of a 2-dynamic coloring of graphs. In the dissertation of Montgomery [40], there is the following conjecture for regular graphs.

Conjecture 5.3.1. *If G is a k-regular graph, then $\chi_2(G) \leq \chi(G) + 2$.*

Note that since $\chi_2(C_5) = 5 = \chi(C_5) + 2$ and $\chi_2(C_n) \leq 4$ for $n \neq 5$, the conjecture is true for $k = 2$, and since $\chi_2(G) \leq \Delta(G) + 1$ except for C_5, and $\chi(G) \leq 2$ if G has an edge, the conjecture is true for $k = 3$. We prove that the inequality in the conjecture holds for every graph with diameter at most 2.

Theorem 5.3.2. *If the diameter of a graph G is at most 2, then $\chi_2(G) - \chi(G) \leq 2$.*

Proof. Consider a minimal counterexample G. Let c be a proper $\chi(G)$-coloring of G, and let S be the set of vertices in G with degree at least 2 not having neighbors with distinct colors under c. Since G is a counterexample, the set S is nonempty. Among the vertices of S, let v be one whose degree in G is smallest. We may assume that v has color 1 and all neighbors of v have color 2.

Since the diameter of G is at most 2, every vertex u not in $N[v]$ has a neighbor in $N(v)$, which implies that u has a neighbor with color 2. Thus, every vertex w in S has color 2 or a neighbor with color 2. Let $S_1 = \{w \in S\colon c(N(w)) = 2\}$, and let $S_2 = \{w \in S\colon c(w) = 2\}$. Since c is a proper coloring, no vertex outside $N(v)$ has color 2, so $S_1 \cap N(v) =$ and $S_2 \subseteq N(v)$. Now, since every vertex in S_1 has a neighbor in $N(v)$ and v has minimum degree among the vertices in S, the vertices in S_1 must have the same neighborhood as v.

Now, if we change the color of one of the neighbors of v, say a, to $\chi(G)+1$, then each vertex of S_1 has neighbors with distinct colors. Now if S_2 is empty, then this yields a 2-dynamic $(\chi(G)+1)$-coloring of G. Let $S_2' = \{x \in S_2\colon x \text{ is not adjacent to } a\}$. If S_2' is not empty, then choose a vertex y with minimum degree in G among the vertices of S_2'. With a similar argument, we conclude that any other vertex in S_2' has the same neighborhood as y. By changing the color of one of the neighbours of y from 1 to $\chi(G)+2$, we obtain a 2-dynamic $(\chi(G)+2)$-coloring of G, which is a contradiction. □

The graph H in Section 5.2 shows that if the diameter of a graph is bigger than 2, then the gap between $\chi(G)$ and $\chi_2(G)$ may be big.

5.4 Cartesian Product

In this section, we study the cartesian product of two graphs. In particular, the r-dynamic chromatic number of the cartesian product of two paths and of two cycles will be investigated. Before computing the numbers, we first prove an upper bound for the r-dynamic chromatic number of the cartesian product of two graphs.

Theorem 5.4.1. *Let G_1 and G_2 be graphs. If $\delta(G_1) \geq r$, then $\chi_r(G_1 \Box G_2) \leq \max\{\chi_r(G_1), \chi(G_2)\}$.*

Proof. Let c_1 be an r-dynamic coloring of G_1 with colors $\{1, \ldots, \chi_r(G_1)\}$, and let c_2 be a proper $\chi(G_2)$-coloring of G_2. Let $M = \max\{\chi_r(G_1), \chi(G_2)\}$. To define a color c on $G_1 \Box G_2$, for $(v_1, v_2) \in V(G_1 \Box G_2)$, let $c((v_1, v_2)) = c_1(v_1) + c_2(v_2) \mod M$. If (v_1, v_2) is adjacent to (v_1', v_2'), then $c_1(v_1) \neq c_1(v_1')$ or $c_2(v_2) \neq c_2(v_2')$, which implies $c(v_1, v_2) \neq c(v_1', v_2')$. Thus, c is a proper coloring.

Furthermore, c is an r-dynamic coloring, since $|c(N((v_1, v_2)))| \geq |c_1(v_1)| \geq \min\{r, d(v_1)\} = r$. Thus, $\chi_r(G_1 \Box G_2) \leq \max\{\chi_r(G_1), \chi(G_2)\}$. $\square$

Other simple graphs we can consider are cartesian products of pathes and cycles. Akbari, Chanbari, and Jahanbekam [2] determined the 2-dynamic chromatic number of $G \Box H$ when $\Delta(H) \leq 2$.

Theorem 5.4.2. *[2] If m and n are at least 2, then $\chi_2(P_m \Box P_n) = 4$. Furthermore,*

$$\chi_2(C_n \Box P_m) = \begin{cases} 3 & 3 \mid n \\ 4 & 3 \nmid n, \text{ and } m \neq 1 \\ \chi_2(C_n) & m = 1 \end{cases}, \quad \text{and} \quad \chi_2(C_m \Box C_n) = \begin{cases} 3 & 3 \mid mn \\ 4 & 3 \nmid mn \end{cases}.$$

Now, we investigate higher dynamic chromatic numbers of such products. When m or n is equal to 1, we already know $\chi_r(P_n \Box P_m)$ from Observation 5.1.2.

Theorem 5.4.3. *If m and n are at least 2, then*

$$\text{for } r \geq 4, \quad \chi_r(P_m \Box P_n) = \begin{cases} 4 & \min\{m, n\} = 2 \\ 5 & \text{otherwise.} \end{cases} \quad \text{and,}$$

$$\chi_3(P_m \Box P_n) = \begin{cases} 4 & \min\{m, n\} = 2 \\ 4 & m \text{ and } n \text{ are both even} \\ 5 & \text{otherwise, except possibly when one is odd and the other is an odd multiple of 2,} \end{cases}$$

Proof. First, we determine $\chi_r(P_m \Box P_n)$ for $r \geq 4$ when $\min\{m, n\} > 2$. If $\min\{m, n\} > 2$, then $\Delta(P_m \Box P_n) = 4$. By Observation 5.1.2, $\chi_r(P_m \Box P_n) = \chi_4(P_m \Box P_n)$, and by Observation 5.1.1, $\chi_4(P_m \Box P_n) \geq 5$. Let $\{(i, j) : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$ be the vertex set of $P_m \Box P_n$. If we define a coloring c on $V(P_m \Box P_n)$ by $c(i, j) = i + 2j \mod 5$, then c is a 4-dynamic 5-coloring of $P_n \Box P_m$, because it is a proper coloring and the neighbors of any vertex have distinct colors. Thus, $\chi_4(P_n \Box P_m) = 5$.

$$c = \begin{array}{cccccc} 0 & 2 & 4 & 1 & 3 & \\ 1 & 3 & 0 & 2 & 4 & \\ 2 & 4 & 1 & 3 & 0 & \cdots \\ 3 & 0 & 2 & 4 & 1 & \\ 4 & 1 & 3 & 0 & 2 & \\ & & \vdots & & & \ddots \end{array}$$

Let g be the function with $g(4k) = 0$, $g(4k+1) = 2$, $g(4k+2) = 1$, and $g(4k+3) = 3$ for all $k \in \mathbb{Z}$. Now, we define a coloring h on $P_m \Box P_n$ by $h(i,j) = g(i) + j \mod 4$ for $i = 0, 1 \mod 4$ and by $h(i,j) = g(i) - j \mod 4$ for $i = 2, 3 \mod 4$.

When $\min\{m, n\} = 2$ and when m and n are both even, the function h is a 3-dynamic proper 4-coloring of $P_m \Box P_n$.

$$h = \begin{array}{cccccc} 0 & 1 & 2 & 3 & 0 & \\ & & & & & \cdots \\ 2 & 3 & 0 & 1 & 2 & \end{array} \qquad \text{when } m = 2$$

$$h = \begin{array}{cccccc} 0 & 1 & 2 & 3 & 0 & 1 \\ 2 & 3 & 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 3 & 2 \end{array} \qquad \text{when } m = 4 \text{ and } n = 6$$

Now, consider the lower bound for this case. We already know that $\chi_3(P_m \Box P_n) = \chi_2(P_m \Box P_n) = 4$ when $m = n = 2$. If $\max\{m, n\} \geq 3$, then $\Delta(P_m \Box P_n) \geq 3$. By Observation 5.1.1, $\chi_3(P_m \Box P_n) \geq 4$. Thus $\chi_3(P_m \Box P_n) = 4$ when $\min\{m, n\} = 2$ and when m and n are both even.

Furthermore, when $\min\{m, n\} = 2$, $\Delta(P_m \Box P_n) = 3$. By observation 5.1.2, $\chi_4(P_m \Box P_n) = \chi_3(P_m \Box P_n) = 4$.

Now, the remaining case is when m or n is not divisible by 2; we need to show that four colors are not enough. Assume to the contrary that there exists a 3-dynamic proper 4-coloring q of $P_m \Box P_n$ when m or n is not divisible by 2. We may assume that n is not divisible by 2. Without loss of generality, we may assume that $q(0,0) = 0$, $q(0,1) = 1$ and $q(1,0) = 2$. Since q is a 3-dynamic proper 4-coloring, the neighbors of any vertex v of degree 3 must have distinct colors different from the color of v. Thus, $q(1,1) = 3$. For the same reason, $q(0,2) = 2$. Similarly, $q(2,0) = 1$. In this fashion, the coloring on the first two rows and the first two columns is determined. For $j \geq 2$ and $i \in \{0,1\}$, we have $q(i,j) = q(1-i, j-2)$.

$$q = \begin{array}{cccccc} 0 & 1 & 2 & 3 & 0 & \cdots \\ 2 & 3 & 0 & 1 & 2 & \cdots \\ 1 & 0 & & & & \\ 3 & 2 & & & & \\ 0 & 1 & & & & \\ \vdots & \vdots & & & & \end{array}$$

Note that colors along the top row cycle through 0,1,2,3, and on the second row, they cycle through 2,3,0,1. Thus, if $n = 4k+1$, then in the first and last column, the ordered pair of colors in the first two rows is $(0,2)$. Also colors down the first column cycle through 0,2,1,3, and in the last column they cycle through 0,2,3,1. The argument we used on the first two rows applies also to the last two rows, so in the first and last columns, the last two rows have the same ordered pair of colors. Since 1 and 3 are flipped in the cyle of colors on the first column and last column, this completes the proof unless $m \equiv 2 \mod 4$ (see figure below).

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 \\ 1 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 3 \\ 3 & 2 & \bullet & \bullet & \bullet & \bullet & \bullet & 2 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 \end{array}$$

When $n = 4k+3$, the ordered pairs in the first two rows of the first and last columns are $(0,2)$ and $(2,0)$. In this case, working down the first and last columns, the same rows have 1 in each, and the same colums have 3 in each. Now the last two rows cannot have the behavior as proved for the first two rows unless $m \equiv 2 \mod 4$. The problem case is again the same.

$\square$

Now, we compute r-dynamic chromatic number of $C_m \square C_n$ for $r \geq 3$. Since $\chi_r(C_m \square C_n) = \chi_r(C_n \square C_m)$, it suffices to check when $m \leq n \mod 4$.

Theorem 5.4.4.

$$\chi_3(C_m \square C_n) \begin{cases} = 4 & \textit{if } m \equiv 0 \textit{ mod4} \;\; \textit{and} \;\; n \equiv t \mod 4 \;\; \textit{for} \;\; t \in \{0,1,2\}, \\ \leq 5 & \textit{if } m \equiv 0 \textit{ mod4} \;\; \textit{and} \;\; n \equiv 3 \mod 4, \;\; \textit{or} \;\; m \equiv 1 \;\; \textit{and} \;\; n \equiv 3 \mod 4, \\ \leq 6 & \textit{otherwise.} \end{cases}$$

Proof. Since $C_m \square C_n$ is 4-regular, $\chi_3(C_m \square C_n) \geq 4$ by Observation 5.1.2.

Case 1. *m and n are both even, but not both congruent to 2 $\mod 4$.*

Let $\{(i,j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$ be the vertex set of $C_m \square C_n$. The function h defined in Theorem 5.4.3 is also a 3-dynamic proper 4-coloring of $C_m \square C_n$ for the congruent classes in this case.

$$A = \begin{matrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{matrix}$$

Case 2. $m \equiv 0 \mod 4$ *and* $n \equiv 1 \mod 4$.

We define h' on $C_m \square C_n$ by $h'(i,j) = h(i,j)$ except when $j = n-1$, and by $h'(i, n-1) = h(i,1)$.

$$\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 1 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & 0 & 3 & 2 & 0 \\ 3 & 2 & 1 & 0 & 2 \end{array}$$

The function h' is a 3-dynamic proper 4-coloring of $C_m \square C_n$ when $m \equiv 0$ and $n \equiv 1 \mod 4$. Thus $\chi_3(C_m \square C_n) = 4$ when $m \equiv 0$ and $n \equiv 1 \mod 4$.

Case 3. $m \equiv 0 \mod 4$ *and* $n \equiv 3 \mod 4$.

We define h'' on $C_m \square C_n$ by $h'(i,j) = h(i,j)$ except when $(i,j) = (1,0)$ and $(i,j) = (m-1, n-1)$, and by $h'(1,0) = h'(m-1,n-1) = 5$.

$$
\begin{array}{cccc|ccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 \\
4 & 3 & 0 & 1 & 2 & 3 & 0 \\
1 & 0 & 3 & 2 & 1 & 0 & 3 \\
3 & 2 & 1 & 0 & 3 & 2 & 4
\end{array}
$$

The function h'' is a 3-dynamic proper 5-coloring of $C_m \square C_n$ when $m \equiv 0$ and $n \equiv 3 \mod 4$, which implies that $\chi_3(C_m \square C_n) \leq 5$.

Case 4. $m, n \equiv 1 \mod 4$.

$$
\begin{array}{cccc|c|cccc|c}
0 & 1 & 2 & 3 & & 0 & 1 & 2 & \underline{4} & \underline{5} \\
\underline{5} & 3 & 0 & 1 & A\cdots & 2 & 3 & 0 & 1 & 3 \\
1 & 0 & 3 & 2 & & 1 & 0 & 3 & 2 & 0 \\
3 & 2 & 1 & 0 & & 3 & 2 & 1 & \underline{5} & \underline{4} \\
\hline
 & & \vdots & & \vdots & & & \vdots & & \vdots \\
\hline
0 & 1 & 2 & 3 & & 0 & 1 & 2 & \underline{4} & \underline{5} \\
\underline{4} & 3 & 0 & 1 & A\cdots & 2 & 3 & 0 & 1 & 3 \\
1 & 0 & 3 & 2 & & 1 & 0 & 3 & 2 & 0 \\
3 & 2 & 1 & 0 & & 3 & 2 & 1 & \underline{5} & \underline{4} \\
\hline
\underline{4} & \underline{5} & 0 & 1 & 2\ 3\ 0\ 1 & 2 & 3 & 0 & 1 & 3
\end{array}
$$

When $m, n \equiv 1 \mod 4$, the above coloring of $(C_m \square C_n)$ is a 3-dynamic proper 6-coloring.

Case 5. $m \equiv 1 \mod 4$ *and* $n \equiv 2 \mod 4$.

$$
\begin{array}{c|c|c|c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \underline{5} & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{array} & A\cdots & \begin{array}{cccc} 0 & 1 & 2 & \underline{5} \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{array} & \begin{array}{cc} 0 & \underline{5} \\ 2 & \underline{6} \\ 1 & 0 \\ 3 & 2 \end{array} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
\begin{array}{cccc} \underline{5} & \underline{6} & 0 & 1 \end{array} & \begin{array}{cccc} 2 & 3 & 0 & 1 \end{array} & \begin{array}{cccc} 2 & 3 & 0 & 1 \end{array} & 1
\end{array}
$$

When $m \equiv 1$, $n \equiv 2 \mod 4$, the above coloring of $(C_m \Box C_n)$ is a 3-dynamic proper 6-coloring.

Case 6. $m \equiv 1 \mod 4$, *and* $n \equiv 3 \mod 4$.

$$
\begin{array}{c|c|c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \underline{5} & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{array} & A\cdots & \begin{array}{ccc} 0 & \underline{5} & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \\ 3 & 2 & \underline{5} \end{array} \\
\hline
\vdots & \vdots & \vdots \\
\hline
\begin{array}{cccc} \underline{5} & 3 & 0 & 1 \end{array} & \begin{array}{cccc} 2 & 3 & 0 & 1 \end{array} & \begin{array}{ccc} 2 & 3 & 0 \end{array}
\end{array}
$$

When $m \equiv 1$, $n \equiv 3 \mod 4$, the above coloring of $(C_m \Box C_n)$ is a 3-dynamic proper 5-coloring.

Case 7. $m \equiv 2 \mod 4$, *and* $n \equiv 2 \mod 4$.

$$
\begin{array}{c|c|c}
\begin{array}{cccc} \underline{6} & 1 & 2 & 3 \\ \underline{5} & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \end{array} & A\cdots & \begin{array}{cc} 0 & 1 \\ 2 & 3 \\ 1 & 0 \\ 3 & 2 \end{array} \\
\hline
\vdots & \vdots & \vdots \\
\hline
\begin{array}{cccc} \underline{6} & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{array} & \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{array} & \begin{array}{cc} 0 & 1 \\ \underline{5} & 3 \end{array}
\end{array}
$$

When $m \equiv 2 \mod 4$ and $n \equiv 2 \mod 4$, the above coloring of $(C_m \Box C_n)$ is a 3-dynamic proper 6-coloring.

Case 8. $m \equiv 2 \mod 4$*, and* $n \equiv 3 \mod 4$.

$$
\begin{array}{cccc|cccc|ccc}
\underline{6} & 1 & 2 & 3 & & & & & 0 & 1 & 2 \\
\underline{5} & 3 & 0 & 1 & \multicolumn{4}{c|}{A\cdots} & 2 & 3 & 0 \\
1 & 0 & 3 & 2 & & & & & 1 & 0 & 3 \\
3 & 2 & 1 & 0 & & & & & 3 & 2 & \underline{5} \\
\hline
\multicolumn{4}{c|}{\vdots} & \multicolumn{4}{c|}{\vdots} & \multicolumn{3}{c}{\vdots} \\
\hline
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 \\
\underline{5} & 3 & 0 & 1 & 2 & 3 & 0 & 1 & \underline{5} & 3 & 0
\end{array}
$$

When $m \equiv 2$, $n \equiv 3 \mod 4$, the above coloring of $(C_m \Box C_n)$ is a 3-dynamic proper 6-coloring.

Case 9. $m, n \equiv 3 \mod 4$.

$$
\begin{array}{cccc|cccc|ccc}
0 & 1 & 2 & 3 & & & & & 0 & 1 & 2 \\
\underline{5} & 3 & 0 & 1 & \multicolumn{4}{c|}{A\cdots} & 2 & 3 & 0 \\
1 & 0 & 3 & 2 & & & & & 1 & 0 & 3 \\
3 & 2 & 1 & 0 & & & & & 3 & 2 & \underline{5} \\
\hline
\multicolumn{4}{c|}{\vdots} & \multicolumn{4}{c|}{\vdots} & \multicolumn{3}{c}{\vdots} \\
\hline
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 \\
\underline{5} & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 \\
1 & 0 & \underline{5} & \underline{6} & \underline{5} & \underline{6} & \underline{5} & \underline{6} & \underline{5} & 0 & 3
\end{array}
$$

When $m \equiv 3 \mod 4$ and $n \equiv 3 \mod 4$, the above coloring of $(C_m \Box C_n)$ is a 3-dynamic proper 6-coloring.

□

Note that $\chi_4(C_m \Box C_n) \geq 5$, since $C_m \Box C_n$ is 4-regular.

Theorem 5.4.5. $\chi_4(C_m \Box C_n) = 5$ *when both of* m *and* n *are divisible by 5.*

Proof. The function c defined in Theorem 5.4.3 is also a 4-dynamic proper 5-coloring of $C_m \Box C_n$ when both of m and n are divisible by 5.

$$c = \begin{array}{cccccc} 0 & 2 & 4 & 1 & 3 & \\ 1 & 3 & 0 & 2 & 4 & \\ 2 & 4 & 1 & 3 & 0 & \cdots \\ 3 & 0 & 2 & 4 & 1 & \\ 4 & 1 & 3 & 0 & 2 & \\ & & \vdots & & & \ddots \end{array}$$

□

Theorem 5.4.6. *$5 \leq \chi_4(C_m \square C_n) \leq 9$ for all m, n.*

Proof. **Case 1.** $m, n \equiv 0 \mod 3$.

$$\begin{array}{ccc|c|ccc} (1,1) & (1,2) & (1,3) & & (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) & & (3,1) & (3,2) & (3,3) \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline (1,1) & (1,2) & (1,3) & & (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) & \cdots & (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) & & (3,1) & (3,2) & (3,3) \end{array}$$

When $m, n \equiv 0 \mod 3$, the above function is a 4-dynamic proper 9-coloring.

Case 2. $m \equiv 0 \mod 3$, *and* $n \equiv 1 \mod 3$.

$$A = \left. \underbrace{\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 1 \end{array}}_{A_1} \right| \underbrace{\begin{array}{ccc} 4 & 5 & 6 \\ 6 & 1 & 2 \\ 2 & 3 & 4 \end{array}}_{A_2}$$

$$B = \begin{array}{c} 7 \\ 8 \\ 9 \end{array}$$

$n \equiv 1 \mod 2$:

$$\frac{m}{3}\left\{\begin{array}{cccccccc} A_1 & A_2 & \ldots & A_1 & A_2 & & A_1 & B \\ \vdots & \vdots & \ldots & \vdots & \vdots & & \vdots & \vdots \\ A_1 & A_2 & \ldots & A_1 & A_2 & & A_1 & B \\ & & & & & & & \\ \multicolumn{5}{c}{\underbrace{\hspace{6em}}_{\frac{n-1}{2}}} & & & \end{array}\right.$$

$n \equiv 0 \mod 2$:

$$\frac{m}{3}\left\{\begin{array}{cccccc} A & A & \ldots & A & & B \\ \vdots & \vdots & \ldots & \vdots & & \vdots \\ A & A & \ldots & A & & B \\ & & & & & \\ \multicolumn{4}{c}{\underbrace{\hspace{4em}}_{\frac{n}{2}}} & & \end{array}\right.$$

When $m \equiv 0$ and $n \equiv 1 \mod 3$, the above function is a 4-dynamic proper 9-coloring of $C_m \square C_n$.

Case 3. $m \equiv 0 \mod 3$ *and* $n \equiv 2 \mod 3$.

$$D = \begin{array}{cc} 7 & 4 \\ 8 & 6 \\ 9 & 2 \end{array}$$

If we replace B in the Case 11 by D, then the above function is a 4-dynamic proper 9-coloring of $C_m \square C_n$ when $m \equiv 0$ and $n \equiv 2 \mod 3$.

Case 5. $m \equiv 1 \mod 3$, *and* $n \equiv 1 \mod 3$.

$$E = \begin{array}{ccc|ccc|c|c} 4 & 8 & 9 & 7 & 8 & 9 & \cdots & 2 \end{array}$$

By attaching E to the last row in Case 11, we have a 4-dynamic proper 9-coloring of $C_m \square C_n$ when $m, n \equiv 1 \mod 3$.

Case 6. $m \equiv 1 \mod 3$ *and* $n \equiv 2 \mod 3$.

$$F = \begin{array}{ccc|ccc|c|cc} 7 & 8 & 9 & 7 & 8 & 9 & \cdots & 5 & 3 \end{array}$$

By attaching F to the last row in Case 12, we have a 4-dynamic proper 9-coloring of $C_m \square C_n$ when $m \equiv 1$ and $n \equiv 2 \mod 3$.

Case 6. $m, n \equiv 2 \mod 3$.

$$G = \begin{array}{cccccc|cccccc|c|cc} 8 & 5 & 6 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & \cdots & 1 & 9 \end{array}$$

By attaching G to the last row in Case 12, we have a 4-dynamic proper 9-coloring of $C_m \square C_n$ when $m \equiv 2$ and $n \equiv 2 \mod 3$.

□

Corollary 5.4.7. $\chi_4(C_3 \square C_3) = 9$.

Proof. In this graph, every two vertices are adjacent to have a common neighbor. Since the graph is 4-regular, in a 4-dynamic coloring, any two vertices must have distinct colors. □

With the Theorem 5.4.5 and Corollary 5.4.7, we note that the bound in Theorem 5.4.6 cannot be improved. In any case of m and n, we may determine $\chi_r(C_m \square C_n)$ with more detailed cases than the proofs in the above Theorems.

References

[1] S. Akbari, M. Ghanbari, S. Jahanbekam, On the Dynamic Chromatic Number of Regular Graphs, manuscript.

[2] S. Akabari, M. Ghanbari, S. Jahanbekam, On the Dynamic Coloring of Cartesian Product Graphs, manuscript.

[3] S. Akbari, M. Ghanbari, S. Jahanbekam, M. Jamaali, List coloring of graphs having cycles of length divisible by a given number, *D*iscrete Appliced Mathematics. **157**, (2009), 3005-300

[4] F. Bäbler, Über die Zerlegung regulärer Streckenkomplexe ungerader Ordnung, *Comment. Math. Helvet.* **10** (1938), 275–287.

[5] C. Berge, Sur le couplagemaximum dun graphe. (French) *C. R. Acad. Sci. Paris* **247**, (1958) 258-259

[6] C. Berge, *Graphs and Hypergraphs*, North Holland, 2nd Edition 1976.

[7] I. Broere, G. Chartrand, O. Oellermann, C.E. Wall, On the edge independence number of a regular graph with large edge connectivity. *C*ombinatorial Mathematics: Proceedings of the Third International Conference, (1985), 94-102

[8] T. Biedl, E.D. Demaine, C.A. Duncan, R. Fleischer, and S.G. Kobourov, Tight bounds on maximal and maximum matchings, *Discrete Math* **285** (2004), 7–15.

[9] A. Brouwer and W. Haemers, Eigenvalues and perfect matchings. *Linear Algebra Appl* **395** (2005), 155-162.

[10] A. Brouwer and W. Haemers, *Spectra of Graphs*, 201pp book (2009), available at `http://homepages.cwi.nl/ ~ aeb/math/ipm.pdf`.

[11] L.W. Beineke, O.R. Oellermann1, R.E. Pippert, The average connectivity of a graph, *Discrete Mathematics* **252** (2002), 31–45 .

[12] S.M. Cioabă, Eigenvalues and edge-connectivity of regular graphs, *Linear Algebra Appl.* **432** (2010) 458–470.

[13] S.M. Cioabă, Perfect matchings, eigenvalues and expansion. *C. R. Math. Acad. Sci. Soc. R. Can.*, **27** (2005), no.4, 101-104.

[14] S.M. Cioabă and D.A. Gregory, Large matchings from eigenvalues, *Linear Algebra and its Applications* **422**, (2007), 308-317.

[15] S.M. Cioabă, D.A. Gregory and W. Haemers, Matchings in regular graphs from eigenvalues, *J.Combin.Theory Ser. B* **99** (2009), 287-297.

[16] G. Chartrand, D.L. Goldsmith, S. Schuster, A sufficient condition for graphs with 1-factors, *Colloq. Math* **41**, (1979), 339344.

[17] G. Chartrand, S.F. Kapoor, L. Lesniak, and S. Schuster, Near 1-factors in graphs, *Proc. Fourteenth Southeastern Conference on Combinatorics, Graph Theory and Computing*, *Congress Numerantium* **41** (1984), 131–147.

[18] V. Chvatal and C. McDiarmid, Small transversals in hypergraphs, *Combinatorica* **12** (1992), 19–26.

[19] S.M. Cioabă, Suil O, Edge-connectivity, eigenvalues, and matchings in regular graphs, *SIAM. J. Disc. Math* **24** (2010) 1470-1481.

[20] S.L. Chandran and S.L. Ram, On the number of minimum cuts in a graph, *SIAM J. Discrete Math.* **18** (2004), no. 1, 177–194.

[21] P. Dankelmann and O.R. Oellermann, On the average connectivity of a graph, *Discrete Appl. Math.* **129** (2003), 305–318.

[22] J. Edmonds, Maximum matching and a polyhedron with $(0,1)$-vertices, *J. Res. Nat. Bur. Standards* **69B** (1965), 125-130.

[23] L. Esperet, F. Kardos, A. King, D. Kral, S. Norine, Exponentially many perfect matchings in cubic graphs, manuscript.

[24] J. Edmonds, L. Lovasz, W.R. Pulleyblank, Brick decompositions and the matching rank of graphs, *Combinatorica* **2** (1982), 247–274.

[25] M. Fiedler, Algebraic connectivity of graphs, *Czech.Math.J* **23** (1973), 298-305.

[26] M. L. Gardner, Forbidden configurations in intersection graphs of r-graphs, *Discrete Mathematics* **31** (1980), 85–88

[27] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.

[28] M.A. Henning, A linear Vizing-like relation relating the size and total domination number of a graph, *Journal of Graph Theory* **49** (2005), 285–290; erratum *Journal of Graph Theory* **54** (2007), 350–353.

[29] M.A. Henning, L. Kang, E. Shan, and A. Yeo, On matching and total domination in graphs, *Discrete Math.* **308** (2008), 2313–2318.

[30] M.A. Henning, A. Soleimanfallah, S. Thomassé, and A. Yeo, Total domination and transversals in hypergraphs.

[31] M. Henzinger and D.P. Williamson, On the number of small cuts, *Information Processing Letters* **59** (1996), 41–44.

[32] M.A. Henning and A. Yeo, Tight lower bounds on the size of a maximum matching in a regular graph, *Graphs and Combinatorics* **23** (2007), 647–657.

[33] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[34] P. Katerinis, Maximum matching in a regular graph of specified connectivity and bounded order, *J. Graph Theory* **11** (1987), 53-58

[35] J. Kim, Suil O, Average connectivity and average edge-connectivity graphs, in preparation.

[36] M. Krivelevich and B. Sudakov, Pseudorandom graphs, in *More sets, graphs and numbers*, 199–262, Bolyai Soc. Math. Stud., 15, Springer, Berlin, 2006.

[37] H.J. Lai, B. Montgomery, and H. Poon, Upper bounds of dynamic chromatic number, *A*rs Combin. **68** (2003), 193-201.

[38] J. Lehel, F. Maffray and M. Preissman, Graphs with largest number of minimum cuts, *Discrete Applied Mathematics* **65** (1996), 387–407.

[39] L. Lovász, *Combinatorial Problems and Exercises*, AMS Chelsea Publishing 2nd Edition 2007.

[40] B. Montgomery, Dynamic Coloring of Graphs, Ph.D Dissertation, Morgantown, West Virginia, 2001.

[41] C. Magnant and D. Martin, A note on the path cover number of regular graphs, *Australas. J. Combin.* **43** (2009), 211–217

[42] D. Naddef, Rank of maximum matchings in a graph, *Math. Programming* **22** (1982), 52–70.

[43] K. Novotny, Maximum matchings in Regular Graphs with Specified Edge-connectivity. *Advanced in Graph Theory* (1991), 285-296.

[44] H. Nagamochi, K. Nishimura and T. Ibaraki, Computing all small cuts in an undirected network, *SIAM J. Discrete Math.* **10** (1997), 469–481.

[45] T. Niessen and B. Randerath, Regular factors of simple regular graphs and the factor spectra, *Discrete Math* **185** (1998), 89103

[46] Suil O, Edge-connectivity and eigenvalues of regular graphs, in preparation.

[47] Suil O, Path cover in regular graphs, in preparation.

[48] Suil O, Number of perfect matchings in regular graphs, in preparation.

[49] Suil O, D.B. West, Balloons, cut-edges, matchings, and total domination in regular graphs of odd degree. *Journal of Graph Theory* **64** (2010) 116-131.

[50] Suil O, D.B. West, Matchings, and edge-connectivity in regular graphs. *European J. Combinatorics* **32** (2011) 324-329.

[51] Suil O, D.B. West, Chinese postman problem in regular Graphs. in preparation.

[52] Suil O, D.B. West, Hehui Wu, Longest Cycles in k-connected Graphs with Given Independence Number, accepted to *J. Combinatorial Theory. B.*

[53] J. Petersen, Die Theorie der regularen Graphen, *Acta Math.* **15** (1891), 193–220.

[54] Ján Plesník, Štefan Znám, On equality of edge-connectivity and minimum degree of a graph.*Arch. Math.(Brno)* **25** (1989), no. 1-2, 19-25.

[55] A. Raspaud, Postman tours and cycle covers, *Discrete Mathematics* **111** (1993), 447-454.

[56] P.D. Seymour, On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.(3)* **38** (1979), no. 3, 423-460

[57] W.T. Tutte, The factorization of linear graphs, *J. Lond. Math. Soc.* **22** (1947), 107–111.

[58] S. Thomassé and A. Yeo, Total domination of graphs and small transversals of hypergraphs, *Combinatorica* **27** (2007), no. 4, 473–487.

[59] M. Voorhoeve: A lower bound for the permanents of certain (0, 1)- matrices, *Nederl. Akad. Wetensch. Indag. Math.* **41** (1979), 83–86.

[60] A. Yeo, Improved bound on the total domination in graphs with minimum degree four, manuscript (2005).